

# JUMPS IN THE ARCHIMEDEAN HEIGHT

PATRICK BROSNNAN AND GREGORY PEARLSTEIN

**ABSTRACT.** We introduce a pairing on local intersection cohomology groups of variations of pure Hodge structure, which we call the asymptotic height pairing. Our original application of this pairing was to answer a question on the Ceresa cycle posed by R. Hain and D. Reed. (This question has since been answered independently by Hain.) Here we apply the pairing to show that a certain analytic line bundle, called the biextension line bundle, defined in terms of normal functions, always extends to any smooth partial compactification of the base. We show that the pairing on intersection cohomology governs the extension of the natural metric on this line bundle studied by Hain and Reed (as well as, more recently, by several other authors). We also prove a positivity property of the asymptotic height pairing, which generalize results of a recent preprint of J. Burgos Gill, D. Holmes and R. de Jong, along with a continuity property of the pairing in the normal function case. Moreover, we show that the asymptotic height pairing arises in a natural way from certain Mumford-Grothendieck biextensions associated to normal functions.

## 1. INTRODUCTION

Let  $\mathcal{H}$  be a weight  $-1$  polarizable variation of pure Hodge structure with integral coefficients over a complex manifold  $S$ . Assume that the local system  $\mathcal{H}_{\mathbb{Z}}$  associated to  $\mathcal{H}$  is torsion-free. The elements of the group  $\mathrm{NF}(S, \mathcal{H})$  of normal functions into  $\mathcal{H}$  can be thought of either as certain holomorphic sections of the Griffiths intermediate Jacobian  $\mathcal{J}(\mathcal{H}) \rightarrow S$ , or as Yoneda extensions of  $\mathbb{Z}$  by  $\mathcal{H}$  in the category  $\mathrm{VMHS}(S)$  of variations of mixed Hodge structure over  $S$ , i.e., elements of  $\mathrm{Ext}_{\mathrm{VMHS}(S)}^1(\mathbb{Z}, \mathcal{H})$ . Define  $\mathrm{NF}(S, \mathcal{H})^{\vee} = \mathrm{Ext}_{\mathrm{VMHS}(S)}^1(\mathcal{H}, \mathbb{Z}(1))$  and note that via duality (cf. Proposition (109))

$$\mathrm{NF}(S, \mathcal{H})^{\vee} \cong \mathrm{NF}(S, \mathcal{H}^{\vee})$$

where  $\mathcal{H}^{\vee} = \mathrm{Hom}(\mathcal{H}, \mathbb{Z}(1))$ .

A biextension variation of type  $\mathcal{H}$  is a variation of mixed Hodge structure  $\mathcal{V}$  over  $S$  equipped with isomorphisms

$$\mathrm{Gr}_0^W(\mathcal{V}) \cong \mathbb{Z}(0), \quad \mathrm{Gr}_{-1}^W(\mathcal{V}) \cong \mathcal{H}, \quad \mathrm{Gr}_{-2}^W(\mathcal{V}) \cong \mathbb{Z}(1). \quad (1)$$

To any biextension variation  $\mathcal{V}$  over  $S$  there is an associated  $C^{\infty}$  function [cf. (39)]

$$h : S \rightarrow \mathbb{R} \quad (2)$$

defined by Deligne's  $\delta$ -splitting (Theorem (2.20), [9]), which measures how far the fibers  $\mathcal{V}_s$  are from being  $\mathbb{R}$ -split mixed Hodge structures.

Given  $\nu \in \mathrm{NF}(S, \mathcal{H})$  and  $\omega \in \mathrm{NF}(S, \mathcal{H})^{\vee}$ , the set of biextension variations of type  $(\nu, \omega)$  consists of all biextension variations over  $S$  such that

$$\begin{aligned} (W_0/W_{-2})\mathcal{V} &\cong \nu; \\ (W_{-1}/W_{-3})\mathcal{V} &\cong \omega. \end{aligned} \quad (3)$$

For any open subset  $U$  of  $S$ ,  $B(\nu, \omega)(U)$  is defined to be the set of isomorphism classes of biextension variations of type  $(\nu, \omega)$  over  $U$ .

Hain and Reed noticed that  $\mathcal{B}(\nu, \omega)$  is naturally an  $\mathcal{O}_S^\times$ -torsor over  $S$ . Therefore the sections of  $\mathcal{B}(\nu, \omega)$  can be identified with the non-vanishing sections of an analytic line bundle  $\mathcal{L} = \mathcal{L}(\nu, \omega)$  over  $S$  which Hain and Reed have called the *biextension line bundle* [19].

In [19], Hain and Reed further observed that  $\mathcal{L}(\nu, \omega)$  comes with a natural metric defined by the requirement that

$$|\mathcal{V}| = \exp(-h(\mathcal{V})) \quad (4)$$

for any  $\mathcal{V} \in \mathcal{B}(\nu, \omega)$ . In the case where  $\nu$  and  $\omega$  arise from a family of homologically trivial algebraic cycles, the function (2) is the archimedean height pairing.

Suppose that  $j : S \hookrightarrow \bar{S}$  is an inclusion of  $S$  as a Zariski open subset of a complex manifold  $\bar{S}$ . Let  $\mathcal{U}$  be a weight  $-1$  polarizable variation of pure Hodge structure with integral coefficients on  $S$ . Assume that  $\mathcal{U}_{\mathbb{Z}}$  is torsion free. Then, M. Saito [34] defined the group of admissible normal functions  $\text{ANF}(S, \mathcal{U})_{\bar{S}}$  to be set of extensions of  $\mathbb{Z}(0)$  by  $\mathcal{U}$  in the category of admissible variations of mixed Hodge structure, i.e.

$$\text{ANF}(S, \mathcal{U})_{\bar{S}} := \text{Ext}_{\text{VMHS}(S)_{\bar{S}}}^1(\mathbb{Z}(0), \mathcal{U})$$

Normal functions of geometric origin (e.g. arising from families of algebraic cycles) are admissible.

*Remark 5.* A choice of polarization  $Q$  of  $\mathcal{U}$  determines a morphism  $a_Q : \mathcal{U} \rightarrow \mathcal{U}^\vee$  given by  $\beta \mapsto Q(\beta, -)$ . This induces a map  $\text{ANF}(S, \mathcal{U})_{\bar{S}} \rightarrow \text{ANF}(S, \mathcal{U}^\vee)_{\bar{S}}$  which we denote by  $\nu \mapsto \nu^\vee$ .

Let  $\nu \in \text{ANF}(S, \mathcal{H})_{\bar{S}}$  and  $\omega \in \text{ANF}(S, \mathcal{H}^\vee)_{\bar{S}}$ . Then, one can ask two questions.

**Q 6.** Does the line bundle  $\mathcal{L}$  extend to a line bundle  $\bar{\mathcal{L}}$  on  $\bar{S}$ ?

**Q 7.** Does the metric  $|\cdot|$  extend to  $\bar{\mathcal{L}}$ ?

Note that, in the (most interesting) case where  $\bar{S}$  is a smooth, projective variety, Q 6 can be rephrased (via the GAGA principle) as asking whether or not  $\mathcal{L}$  is algebraic. In Theorem 194 we show that Q 6 has a positive answer. While there are, in general, many extensions of  $\mathcal{L}$  to  $\bar{S}$ , there is a canonical choice of extension  $\bar{\mathcal{L}}_{\text{can}} \in \text{Pic } \bar{S} \otimes \mathbb{Q}$ .

On the other hand, we will give a criterion in terms of the local intersection cohomology of  $\nu$  and  $\omega$  for Q 7 to have a negative answer and, using this criterion, we will give an example where the metric does not extend.

More precisely, let  $\bar{S} = \Delta^r$  be a polydisk with local coordinates  $(s_1, \dots, s_r)$  and  $S = \Delta^{*r}$  be the complement of the divisor  $s_1 \dots s_r = 0$ . Let  $\mathcal{H}$  be a torsion free variation on  $S$  with unipotent monodromy and  $\text{IH}^1(\mathcal{H})$  denote the local intersection cohomology group attached to  $\mathcal{H}$ . If  $i : \{0\} \rightarrow \Delta^r$  is the inclusion of the origin, then, in terms of the intersection complex  $\text{IC}(\mathcal{H})$  on  $\Delta^r$  associated to  $\mathcal{H}$ ,  $\text{IH}^1(\mathcal{H}) = H^{-r+1}(i^* \text{IC}(\mathcal{H}))$ . The group  $\text{IH}^1(\mathcal{H})$  is a subgroup of  $H^1(\Delta^{*r}, \mathcal{H})$ . Moreover, since  $\mathcal{H}$  is a variation of weight  $-1$ , the natural map (cf. Theorem (108))

$$\text{sing} : \text{ANF}(\Delta^{*r}, \mathcal{H})_{\Delta^r} \rightarrow H^1(\Delta^{*r}, \mathcal{H})$$

factors through  $\text{IH}^1(\Delta^r, \mathcal{H})$  [3]. We then define, for each  $t \in \mathbb{Q}_{\geq 0}^r$  a pairing

$$h(t) : \text{IH}^1(\mathcal{H}) \otimes \text{IH}^1(\mathcal{H}^*) \rightarrow \mathbb{Q}, \quad (8)$$

which we call the asymptotic height pairing.

We will show that the pairing  $h$  has the following properties (cf. Theorem (133), Proposition (134) and Theorem 161).

- (i) It is homogeneous of degree 1 in  $t$ :  $h(\lambda t)(\alpha, \beta) = \lambda h(t)(\alpha, \beta)$  for  $\lambda \in \mathbb{Q}_{\geq 0}$ .
- (ii) If  $(\nu, \omega) \in \text{ANF}(\Delta^{*r}, \mathcal{H}) \times \text{ANF}(\Delta^{*r}, \mathcal{H}^\vee)$ , then  $h(\nu, \omega)(t)$  is a rational function which extends continuously to the cone  $\mathbb{R}_{\geq 0}^r$ .

- (iii) For each fixed  $t$ ,  $h(t)$  induces a morphism of Hodge structures with  $\mathbb{Q}$  given the pure Hodge structure of weight 0. Thus  $h(t)$  factors through a pairing

$$\bar{h}(t) : \mathrm{Gr}_0^W \mathrm{IH}^1(\mathcal{H}) \otimes \mathrm{Gr}_0^W \mathrm{IH}^1(\mathcal{H}^*) \rightarrow \mathbb{Q}(0), \quad (9)$$

which is a morphism of Hodge structures.

- (iv) Suppose  $\phi : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathbb{Q}(1)$  is a polarization. Then, for  $t \in \mathbb{Q}_+^*$ , the pairing on  $\mathrm{Gr}_0^W \mathrm{IH}^1(\mathcal{H})$  induced by  $\phi$  and (9) is a polarization.

*Remark 10.* In section 7 we show that  $h(t)$  can be generalized to a pairing (145) on the higher intersection cohomology groups  $\mathrm{IH}^p(\mathcal{H})$  as well.

Returning to Q 7, it follows from Theorem (20) below that the metric  $|\cdot|$  on  $\mathcal{L}$  does not extend from  $S$  to  $\bar{S}$  if  $h(t)$  is not linear in  $t$ .

**Corollary 11.**  $h(t)(v, v^\vee) \geq 0$  for all  $v, t$ . Moreover,  $h(t)(v, v^\vee) = 0$  for all  $t$  if and only if  $\mathrm{sing}(v) = 0$ .

*Proof.*  $\mathrm{IH}^1(\mathcal{H})$  carries a mixed Hodge structure such that  $W_0 \mathrm{IH}^1(\mathcal{H}) = \mathrm{IH}^1(\mathcal{H})$ . Furthermore, the map  $\mathrm{sing} : \mathrm{ANF}(\Delta^{*r}, \mathcal{H})_{\Delta^r} \rightarrow \mathrm{IH}^1(\mathcal{H})$  factors through the subgroup  $\mathrm{Hdg} \mathrm{IH}^1(\mathcal{H})$  consisting of classes in the image of morphisms in  $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(0), \mathrm{IH}^1(\mathcal{H}))$ . Therefore (cf. (9)), it follows from (iii) and (iv) above that  $h(t)(v, v^\vee) \geq 0$ . Moreover,  $h(t)(v, v^\vee) = 0$  for all  $t$  if and only if  $\mathrm{sing}(v) = 0$ .  $\square$

Singularities of normal functions can also be defined in the non-normal crossing case [3]. Let  $S$  is a complex manifold and  $j : S \hookrightarrow \bar{S}$  is an inclusion of  $S$  as a Zariski open subset of a complex manifold  $\bar{S}$ . We say that  $v \in \mathrm{ANF}(S, \mathcal{H})_{\bar{S}}$  is *singular* at  $p \in \bar{S} - S$  if  $\mathrm{sing}_p(v) \neq 0$ .

### 1.1. Applications.

**Griffiths–Green Program.** Let  $X$  be a complex projective manifold of dimension  $2n$  and  $L \rightarrow X$  be a very ample line bundle. Let  $\bar{P} = |L| = \mathbb{P}(H^0(X, L))$ , and let  $\mathcal{X} \subset X \times \bar{P}$  be the incidence variety consisting of pairs  $(x, \sigma) \in X \times \bar{P}$  such that  $\sigma(x) = 0$ . Let  $\pi : \mathcal{X} \rightarrow \bar{P}$  denote the projection map  $\pi(x, \sigma) = \sigma$ . For  $\sigma \in \bar{P}$  let  $X_\sigma = \pi^{-1}(\sigma)$ . Let  $P \subset \bar{P}$  denote the locus of points over which  $\pi$  is smooth, and  $\hat{X} = \bar{P} - P$ .

Let  $\zeta$  be a primitive integral, non-torsion Hodge class on  $X$  of type  $(n, n)$ . Then, via Deligne cohomology,  $\zeta$  determines an admissible normal function

$$v_\zeta : P \rightarrow J(\mathcal{H})$$

where  $\mathcal{H}$  is the variation of Hodge structure on the variable cohomology of the smooth hyperplane sections of  $X$ .

**Conjecture 12.** For every triple  $(X, L, \zeta)$  as above, there exists an integer  $d > 0$  such that after replacing  $L$  by  $L^d$ , the normal function  $v_\zeta$  is singular at some point of  $\hat{X}$ .

**Theorem 13.** [16, 3, 12] The Hodge conjecture is equivalent to Conjecture (12).

By Theorem (7.18) of [3], Conjecture (12) is equivalent to the following statement:

**Conjecture 14.** For every triple  $(X, L, \zeta)$  as above, there exists  $d > 0$  and a resolution of singularities  $\bar{S} \rightarrow \bar{P}$  of  $\hat{X}$  for  $\bar{P} = |L^d|$  such that  $f^*v_\zeta$  is singular at some point of  $\bar{S} - S$ .

So, using Corollary 11, the Hodge conjecture would imply that there exists an  $\bar{S}$  as above such that the biextension metric on  $\mathcal{L}(v_\zeta, v_\zeta^\vee)$  does not extend to  $\bar{\mathcal{L}}(v_\zeta, v_\zeta^\vee)$ .

**Ceresa Cycle.** Let  $C$  be a smooth projective curve of genus  $g > 2$ . Pick a point  $p \in C$ . Then, the maps  $x \mapsto x - p$  and  $x \mapsto p - x$  give two embeddings of  $C$  into  $\text{Jac}(C)$ . The difference, denoted  $C - C^-$ , is a homologically trivial algebraic 1-cycle on  $\text{Jac}(C)$  which has a well defined image in the intermediate Jacobian of the primitive part of the cohomology of  $\text{Jac}(C)$ , independent of  $p$ .

Let  $\mathcal{M}_g$  denote the moduli space of smooth projective curves of genus  $g > 2$ . Then, application of the previous construction determines a normal function  $\nu = \nu_{C-C^-}$  over  $\mathcal{M}_g$ . In Theorem (8.3) of [20], Richard Hain showed that is the “basic” normal function over  $\mathcal{M}_g$ .

In [19], Hain and Reed showed explicitly that  $\mathcal{L}(\nu, \nu^\vee)$  extended to a line bundle over  $\overline{\mathcal{M}}_g$ , and asked if the metric also extends to  $\overline{\mathcal{M}}_g$ . Our calculations show that it does not. (Hain has since given an independent proof of this result. See [18].)

## 1.2. Existence and Regularity.

**Existence of  $\mathcal{L}$ , Local Case.** In the case where  $\mathcal{H}$  is a torsion free variation of pure Hodge structure of weight  $-1$  over a punctured disk  $\Delta^{*r}$ , it follows from the results of §3 that  $\mathcal{L}$  has an analytic extension over  $\Delta^r$ .

Moreover, assuming the monodromy of  $\mathcal{H}$  is unipotent, the biextension metric has a continuous extension on the complement of a codimension 2 subset after a suitable renormalization. In the case where  $\omega = \nu^\vee$  the biextension metric has a plurisubharmonic extension as described below. More precisely, given a biextension variation  $\mathcal{V}$  such that  $Gr_{-1}^W(\mathcal{V}) = \mathcal{H}$  has unipotent monodromy, it follows from (Theorem (5.37) [31]) that there exists a unique element

$$\mu(\mathcal{V}) \in V_r \quad (15)$$

with the following property:

**Property 16.** Let  $\phi : \Delta \rightarrow \Delta^r$  be a holomorphic map such that  $\phi(0) = 0$  and  $\phi(\Delta^*) \subset \Delta^{*r}$ . Suppose that  $\phi(s) = (\phi_1(s), \dots, \phi_r(s))$  with  $\phi_i(s) = a_i s^{m_i} + \text{higher order terms}$  and  $m_j > 0$ . Then

$$h(V_{\phi(s)}) \sim -\mu(\mathcal{V})(m_1, \dots, m_r) \log |s| \quad (17)$$

for  $s$  close to 0. (In other words, the difference between the left side and the right is bounded near 0.)

*Remark 18.* The value of  $\mu(m)$  depends only on the local monodromy logarithm  $N(m) = \sum_j m_j N_j$ , the weight filtration  $W(V_Q)$  and the choice of positive generators  $1 \in Gr_0^W(V_Q)$  and  $1^\vee \in Gr_{-2}^W(V_Q)$ . More precisely, as long as  $m_1, \dots, m_r \geq 0$ , the admissibility of  $\mathcal{V}$  implies the existence of a rational direct sum decomposition

$$V_Q = \mathbb{Q}v_0 \oplus U_Q \oplus \mathbb{Q}v_{-2} \quad (19)$$

such that

- $v_0$  projects to  $1 \in Gr_0^W$  and  $N(m)(v_0) \in W_{-2}$ ;
- $U_Q$  is an  $N(m)$ -invariant subspace of  $W_{-1}$ ;
- $v_{-2} \in W_{-2}$  projects to  $1^\vee$ .

Then, by Theorem (5.19) of [31], for any such decomposition (19) we have  $N(m)v_0 = \mu(m)v_{-2}$ . Indeed, in the language of [31], such a decomposition of  $V_Q$  defines a grading  $Y'$  of  $W$  (cf. §2) such that  $[Y', N]$  lowers  $W$  by 2. For future use, we define  $\mu_j = \mu(\epsilon_j)$  where  $\epsilon_j$  is the  $j$ 'th unit vector.

**Theorem 20.** If  $\mathcal{V}$  is a biextension variation of type  $(\nu, \omega)$  over  $\Delta^{*r}$  with unipotent monodromy then  $h(m)(\nu, \omega) = \sum_j m_j \mu_j - \mu(\mathcal{V})$ .

*Proof.* See Theorem (153) □

**Regularity of the Biextension Metric.** Let  $\mathcal{V}$  be a biextension variation with unipotent monodromy defined on the complement of the divisor  $D = \{s_1 \cdots s_k = 0\}$  in a polydisk  $\Delta^r$  with local coordinates  $(s_1, \dots, s_r)$ . In this case, it follows by Theorem (5.8) of [22] that  $\log |\mathcal{V}|$  is locally  $L^1$ .

Let  $E$  denote the singular locus of  $D$  and  $D_j \subset D$  denote the divisor defined by  $s_j = 0$ . Given a point  $p \in (D - E) \cap D_j$  let  $\Delta(p) \subset \Delta^r$  denote the disk defined by  $s_i = s_i(p)$  for  $i \neq j$ . Then,  $\mathcal{V}$  restricts to an admissible biextension variation on  $\Delta^*(p) = \Delta(p) - \{p\}$  with unipotent monodromy, and hence

$$h(\mathcal{V}|_{\Delta^*(p)}) \sim -\mu_j \log |s_j|$$

in the notation of Remark (18).

Define

$$\bar{h}(\mathcal{V}) = h(\mathcal{V}) + \sum_{j=1}^k \mu_j \log |s_j| : \Delta^r - D \rightarrow [-\infty, \infty) \quad (21)$$

Then, for any point  $p \in D - E$ , by restriction to a small polydisk containing  $p$ , we can reduce the study of the regularity of  $h(\mathcal{V})$  at  $p$  to the case  $k = 1$ .

Assume therefore that  $k = 1$  and  $\mathcal{V}$  has unipotent monodromy about  $s_1 = 0$ .

**Theorem 22.** *Let  $\mathcal{V}$  be a biextension variation with unipotent monodromy defined on the complement of the divisor  $s_1 = 0$  in  $\Delta^r$ . Then,  $\bar{h}(\mathcal{V})$  extends continuously to  $\Delta^r$ .*

*Proof.* See Theorem (77). □

*Remark 23.* The proof of Theorem (22) involves showing that in the case of a biextension over  $\Delta^*$

$$\lim_{s \rightarrow 0} h(\mathcal{V}) + \mu \log(s) = ht(N, F, W)$$

where  $(e^{zN}F, W)$  is the associated nilpotent orbit. See Theorem (74) for details.

Returning now to the general case  $\Delta^r - D$  where  $D$  is given by the vanishing of  $s_1 \cdots s_k = 0$  we first recall the following result in the case where  $\omega = v^\vee$ :

**Theorem 24.** [30] *If  $\mathcal{V}$  is a biextension variation of type  $(v, v^\vee)$  on a complex manifold  $S$  then  $h(\mathcal{V})$  is plurisubharmonic function on  $S$ .*

To continue, we recall the following [See (7.1) and (7.2) [28]]: Let  $\{f_j\}$  be a sequence of plurisubharmonic functions on a domain  $\Omega \subseteq \mathbb{C}^r$  which are locally bounded from above. Given  $z \in \Omega$  let

$$f(z) = \limsup_{j \rightarrow \infty} f_j(z)$$

and define  $f^*(z) = \limsup_{w \rightarrow z} f(w)$ . Then,  $f^*$  is plurisubharmonic.

**Theorem 25.** *Let  $\mathcal{V}$  be a biextension of type  $(v, v^\vee)$  with unipotent monodromy on the complement of  $D = \{s_1 \cdots s_k = 0\}$  in the polydisk  $\Delta^r$ . Suppose that  $\mu_1, \dots, \mu_k \geq 0$ . Then,  $\bar{h}(\mathcal{V})$  has a unique plurisubharmonic extension to  $\Delta^r$ .*

*Proof.* By Theorem (22) it follows that  $g = \bar{h}(\mathcal{V}) \in C^0(\Delta^r - D)$  extends to  $\bar{g} \in C^0(\Delta^r - E)$ . Moreover, since  $h(\mathcal{V})$  is plurisubharmonic on  $\Delta^r - D$  and all  $\mu_j \geq 0$  by assumption, it follows that  $g$  is plurisubharmonic on  $\Delta^r - D$ . In the paragraphs below, we shall prove that  $\bar{g}$  is plurisubharmonic on  $\Delta^r - E$ . Since  $E$  has codimension 2, it then follows from a theorem of Grauert and Remmert[15] that  $\bar{g}$  has a unique plurisubharmonic extension to  $\Delta^r$ .

To prove that  $\bar{g}$  is plurisubharmonic on  $\Delta^r - E$ , consider the sequence of functions

$$f_j = \bar{g} + \frac{1}{j} \log |s_1 \cdots s_k| : \Delta^r - E \rightarrow [-\infty, \infty)$$

where  $f_j = -\infty$  on  $D - E$ . Clearly, each  $f_j$  is plurisubharmonic on  $\Delta^r - E$  since

- $g$  and  $\log |s_1 \dots s_k|$  are plurisubharmonic on  $\Delta^r - D$ ;
- $f_j$  is upper semicontinuous on  $\Delta^r - E$  as the sum of the continuous function  $\bar{g}$  and the upper semicontinuous function  $\frac{1}{j} \log |s_1 \dots s_k|$ .
- $f_j$  has the subaveraging property at any point of  $D - E$ ;

Moreover, since  $\bar{g}$  is continuous and all  $\mu_j \geq 0$  it follows that each  $f_j$  is locally bounded above. Accordingly,  $f^*$  is plurisubharmonic. Finally, since  $\bar{g}$  is continuous on  $\Delta^r - E$  it follows from the definition of the sequence  $f_j$  that  $f^* = \bar{g}$ .  $\square$

Having constructed an extension of  $\bar{h}(\mathcal{V})$  to  $\Delta^r$ , we can inquire as to its value at  $(0, \dots, 0) \in \Delta^r$ . To this end, let  $m = (m_1, \dots, m_r) \in \mathbb{Z}_{>0}^r$  and  $\phi : \Delta \rightarrow \Delta^r$  be the test curve defined by  $s_j = s^{m_j}$ ,  $j = 1, \dots, r$ . Then, by Theorem 20

$$\begin{aligned} \phi^*(\bar{h}(\mathcal{V})) &= \phi^*(h(\mathcal{V})) + \left( \sum_j m_j \mu_j \right) \log |s| \sim (-\mu(\mathcal{V})(m) + \sum_j m_j \mu_j) \log |s| \\ &\sim h(m)(v, v^\vee) \log |s| \end{aligned}$$

By Corollary (11), we have the following two cases:

- (a) If  $\text{sing}_0(v) \neq 0$  then  $h(m) > 0$  and hence  $\lim_{s \rightarrow 0} \phi^*(\bar{h}(\mathcal{V})) = -\infty$ .
- (b) If  $\text{sing}_0(v) = 0$  then  $h(m) = 0$  and hence  $\phi^*(\bar{h}(\mathcal{V}))$  is bounded near zero. Since  $\bar{h}(\mathcal{V})$  plurisubharmonic it follows from upper semicontinuity that  $\bar{h}(\mathcal{V})$  is bounded near zero.

**1.3. Mumford-Grothendieck Biextensions.** There is a close relationship between the concept of biextension variation from (1) and the concept of a biextension introduced by Mumford and studied extensively by Grothendieck in [36]. Essentially, as we vary the normal functions  $v$  and  $\omega$ , the set of all biextension variations  $\mathcal{V}$  forms a biextension (in the Mumford-Grothendieck sense) of  $\text{NF}(S, \mathcal{H}) \times \text{NF}(S, \mathcal{H}^\vee)$  by the sheaf  $\mathcal{O}_S^\times$ . (See Corollary 175.) Starting in §10 we exploit this connection, and we use it in §12 to define another pairing on intersection cohomology with values in  $\mathbb{Q}/\mathbb{Z}$  which we use to answer Question 6.

Unfortunately, the two terminologies clash. We think that we have solved the problems by writing the sections involving Mumford-Grothendieck biextensions in such a way that it is always clear which type of biextension we mean. But we would like to warn the reader (and apologize for the fact) that it will sometimes be decide this based on context.

The Mumford-Grothendieck biextensions we deal with come out of another notion studied by Grothendieck in [36], the notion of *extensions panachées*. These are objects  $X$  in an abelian category equipped with a two step filtration. For example, biextension variations are *extensions panachées* of  $\mathbb{Z}$  by  $\mathcal{H}$  by  $\mathbb{Z}(1)$ . In §10, we work out the (slightly subtle) conditions under which the set of isomorphism classes of *extension panachées* give rise to a Mumford-Grothendieck biextension. Then we apply these to certain Mumford-Grothendieck biextensions, mainly of topological type, which arise in connection with the asymptotic height pairing. In §15, we use these results, to generalize the asymptotic height pairing from the case of variations with unipotent monodromy in the normal crossing case to arbitrary pure variations on  $S$  in a neighborhood of some  $s \in \bar{S}$  (where  $S$  is a Zariski open subset of a complex manifold  $\bar{S}$ ).

In this paper, we translate the term *extensions panachées* as “mixed extensions.” The (probably better) term “blended extensions” is used by D. Bertrand in the abstract to [2].

**1.4. Related Work.** The thesis [11] of D. Lear contains the initial results of the asymptotics of the height over a punctured disk. An explicit example of the non-linearity of  $\mu$  appears in Example (5.44) of [31]. The asymptotics of the biextension bundle also appear in the recent work of J.

Burgos Gill, D. Holmes and R. de Jong [7] and [6]. Another approach to limits of heights having to do with Feynman amplitudes is given in [1]. A preliminary discussion of heights from the viewpoint of log geometry and compactification of mixed period domains may be found at the end of [27], with additional results appearing in [26].

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## 2. VARIATIONS OF MIXED HODGE STRUCTURE

**Weights.** Suppose  $M$  is any object with a weight filtration  $W$ . For example,  $M$  could be a mixed Hodge structure, a variation of mixed Hodge structure or a mixed Hodge module. We say that  $M$  has *weights in an interval*  $[a, b]$  if  $\text{Gr}_k^W M = 0$  for  $k \notin [a, b]$ .

**Mixed Hodge Structures.** In this paper, all mixed Hodge structure are defined over  $A = \mathbb{Z}, \mathbb{Q}$  or  $\mathbb{R}$ . We let  $\mathbb{F} = \mathbb{Q}$  or  $\mathbb{R}$ . For an integer  $n$ , we let  $A(n)$  denote the pure  $A$ -Hodge structure of weight  $-2n$  with underlying  $A$ -module  $A$  [not  $(2\pi i)^n A$ ]. If  $H$  and  $K$  are mixed Hodge structures and  $n \in \mathbb{Z}$ , an element  $f \in \text{Hom}_{\text{MHS}}(H, K(-n))$  is called a  $(n, n)$  morphism of Hodge structures. A morphism of mixed Hodge structures is a  $(0, 0)$  morphism.

A mixed Hodge structure  $(F, W)$  induces a unique, functorial bigrading [Theorem (2.13), [9]]

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q} \quad (26)$$

on the underlying complex vector space  $V_{\mathbb{C}}$  such that

- (a)  $F^p = \bigoplus_{a \geq p} I^{a,b}$ ;
- (b)  $W_k = \bigoplus_{a+b \leq k} I^{a,b}$ ;
- (c)  $I^{\bar{p},q} = I^{q,p} \bmod \bigoplus_{a < q, b < p} I^{a,b}$ .

A mixed Hodge structure is said to be split over  $\mathbb{R}$  if  $\bar{I}^{p,q} = I^{q,p}$ . In this case

$$I^{p,q} = F^p \cap \bar{F}^q \cap W_{p+q} \quad (27)$$

If  $K$  is a field of characteristic zero and  $W$  is an increasing filtration of a finite dimensional  $K$ -vector space  $V$  then a grading of  $W$  is a semisimple endomorphism  $Y$  of  $V$  with integral eigenvalues such that

$$W_k = E_k(Y) \bigoplus W_{k-1}$$

for each index  $k$ , where  $E_k(Y)$  denotes the  $k$ -eigenspace of  $Y$ . In particular, a mixed Hodge structure  $(F, W)$  determines a  $\mathbb{C}$ -grading  $Y = Y_{(F,W)}$  of  $W$  by the rule

$$E_k(Y) = \bigoplus_{p+q=k} I^{p,q} \quad (28)$$

In what follows, the grading  $Y_{(F,W)}$  will be called the Deligne grading of  $(F, W)$ .

The set of all gradings of  $W$  will be denoted  $\mathcal{Y}(W)$ . The subalgebra of  $gl(V)$  consisting of all elements  $\alpha$  such that

$$\alpha(W_k) \subseteq W_{k-\ell}$$

for all  $k$  will be denoted  $W_{-\ell}gl(V)$ .

**Proposition 29** (Prop. (2.2) [9]).  $\exp(W_{-1}gl(V))$  acts simply transitively on  $\mathcal{Y}(W)$  by the adjoint action.

**Proposition 30.** A mixed Hodge structure  $(F, W)$  is split over  $\mathbb{R}$  if and only if  $Y = \bar{Y}$  where  $Y$  is the Deligne grading of  $(F, W)$ .

*Proof.* If  $(F, W)$  is  $\mathbb{R}$ -split then  $Y = \bar{Y}$  since  $\bar{I}^{p,q} = I^{q,p}$ . Conversely, suppose that  $Y = \bar{Y}$  and let  $\oplus_{p+q=k} H^{p,q}$  be the Hodge decomposition of  $Gr_k^W$ . Define

$$J^{p,q} = \{v \in E_{p+q}(Y) \mid [v] \in H^{p,q} \subseteq Gr_{p+q}^W\}$$

Then,  $V_{\mathbb{C}} = \oplus_{p,q} J^{p,q}$  satisfies conditions (a)–(c) of (26): Condition (a) follows from the fact that  $Y$  is a grading of  $W$  which preserves  $F$ . Condition (b) follows from the fact that  $Y$  is a grading of  $W$ . Finally, condition (c) follows from the fact that  $\bar{J}^{p,q} = J^{q,p}$ . Thus,  $I^{p,q} = J^{p,q}$  by the uniqueness of (26), and hence  $(F, W)$  is split over  $\mathbb{R}$ .  $\square$

If  $(F, W)$  is a mixed Hodge structure with underlying complex vector space  $V_{\mathbb{C}}$  then

$$\Lambda_{(F,W)}^{-1,-1} = \bigoplus_{p,q < 0} gl(V_{\mathbb{C}})^{p,q} \quad (31)$$

Note that by property (c) of (26),  $\Lambda^{-1,-1}$  is closed under complex conjugation. It follows from the defining properties (a)–(c) of (26) that

$$e^{\lambda} I_{(F,W)}^{p,q} = I_{(e^{\lambda} F, W)}^{p,q} \quad (32)$$

**Theorem 33** (Theorem (2.20) [9]). Given a mixed Hodge structure  $(F, W)$  with underlying real vector space  $V_{\mathbb{R}}$  there exists a unique real element

$$\delta \in \Lambda_{(F,W)}^{-1,-1} \cap gl(V_{\mathbb{R}}) \quad (34)$$

such that  $(e^{-i\delta} F, W)$  is a mixed Hodge structure which is split over  $\mathbb{R}$ . Moreover,  $\delta$  commutes with all  $(r, r)$ -morphisms of  $(F, W)$ .

The splitting (34) is henceforth called the Deligne splitting or  $\delta$ -splitting of  $(F, W)$ . The defining equation (ibid) for  $\delta$  is

$$\overline{Y_{(F,W)}} = e^{-2i \text{ad } \delta} Y_{(F,W)} \quad (35)$$

**Definition 36.** An unoriented biextension in the category of  $A$ -mixed Hodge structures is an  $A$ -mixed Hodge structure  $(F, W)$  with weights in  $[-2, 0]$  such that  $Gr_0^W, Gr_{-2}^W$  have rank 1. A biextension mixed Hodge structure is an unoriented biextension equipped with a choice of isomorphisms  $Gr_0^W \cong A(0)$  and  $Gr_{-2}^W \cong A(1)$ . The positive generators of  $A(0)$  and  $A(1)$  will be denoted 1 and  $1^{\vee}$  respectively.

If  $(F, W)$  is a biextension mixed Hodge structure then  $\eta$  is the unique  $(-1, -1)$ -morphism of  $(F, W)$  such that the induced map  $Gr_0^W \rightarrow Gr_{-2}^W$  sends 1 to  $1^{\vee}$ . The group  $(\mathbb{C}, +)$  acts additively on the set of biextension mixed Hodge structures by the rule

$$t + (F, W) = (e^{t\eta} F, W) \quad (37)$$

Since  $\eta$  is a  $(-1, -1)$ -morphism,  $t + (F, W) = (F, W)$  if and only if  $t = 0$ .



**Definition 38.** The height of a biextension mixed Hodge structure  $(F, W)$  is the unique real number  $h(F, W)$  such that

$$2\pi\delta_{(F,W)} = h(F, W)\eta \quad (39)$$

**Lemma 40.** Let  $(F, W)$  be a biextension mixed Hodge structure. Then,

(a) For any complex number  $t$ ,

$$h(e^{t\eta}F, W) = h(F, W) + 2\pi\text{Im}(t)\eta \quad (41)$$

(b)  $h(F, W)$  depends only the underlying isomorphism class of  $(F, W)$  as an  $\mathbb{R}$ -biextension, i.e.  $h(F, W)$  is invariant under isomorphisms of  $\mathbb{R}$ -mixed Hodge structures such that the induced maps

$$Gr_0^W \rightarrow Gr_0^W, \quad Gr_{-2}^W \rightarrow Gr_{-2}^W$$

are the identity.

(c)  $h(F, W) = 0$  if and only if  $(F, W)$  is  $\mathbb{R}$ -split.

*Proof.* Item (a) follows by direct computation from equations (35) and (37). Item (b) follows from the fact that  $\delta$  is an isomorphism invariant of  $\mathbb{R}$ -mixed Hodge structures, i.e. an isomorphism of  $\mathbb{R}$ -mixed Hodge structures induces an isomorphism of Deligne bigradings, and hence intertwines  $\delta$ -invariants. Regarding item (c), observe that by construction  $\delta_{(F,W)}$  vanishes if and only if  $(F, W)$  is split over  $\mathbb{R}$ .  $\square$

In the case where  $X$  is a smooth complex projective variety of dimension  $n$  and  $Z$  and  $W$  are null homologous cycles of dimensions  $d$  and  $e$  such that  $|Z| \cap |W|$  is empty and  $d + e = n - 1$  there exists a canonical subquotient  $(F, W)$  of the mixed Hodge structure on  $H_{2d+1}(X - |W|, |Z|)$  such that  $h(F, W)$  is the archimedean height of the pair  $(Z, W)$ . See [17] and equation (5.4) of [31] for details.

If  $(F, W)$  is a biextension mixed Hodge structure let  $\tilde{B} = \tilde{B}(F, W)$  denote the  $(\mathbb{C}, +)$ -orbit of  $(F, W)$  under (37). Since  $\eta$  is a  $(-1, -1)$ -morphism of  $(F, W)$ , any two elements of  $\tilde{B}(F, W)$  induce the same mixed Hodge structures on  $Gr_k^W$  as well as  $W_0/W_{-2}$  and  $W_{-1}$ . Let  $B$  be the quotient of  $\tilde{B}$  by  $(A, +) \subset (\mathbb{C}, +)$ . By (41), the height function (39) descends to a function  $h : B \rightarrow \mathbb{R}$ . For  $b \in B$  we define

$$|b| = \exp(-h(b)) \quad (42)$$

In the case  $A = \mathbb{Z}$ , the set  $B$  has a simply transitive  $\mathbb{C}^*$  action given by

$$t \cdot [(\tilde{F}, W)] = [(e^{\frac{1}{2\pi i} \log(t)\eta} \tilde{F}, W)] \quad (43)$$

since  $\frac{1}{2\pi i} \log(t)$  is well defined modulo  $\mathbb{Z}$ .

**Proposition 44.** Let  $(F, W)$  be a  $\mathbb{Z}$ -biextension mixed Hodge structure and  $t \in \mathbb{C}^*$ . Then,  $|t \cdot b| = |t||b|$ .

*Proof.* By (41),  $h(t \cdot b) = h(b) - \log |t|\eta$ .  $\square$

**Classifying Spaces.** A polarization of a pure  $A$ -Hodge structure  $H$  of weight  $k$  is a  $(-1)^k$ -symmetric, non-degenerate bilinear form

$$Q : H \otimes H \rightarrow A(-k)$$

which is a morphism of Hodge structure such that if  $C$  is the Weil operator which acts on  $H^{p,q}$  as  $i^{p-q}$  then  $\langle u, v \rangle = Q(Cu, \bar{v})$  is a positive definite hermitian inner product on  $H_{\mathbb{C}}$ .

A graded-polarization of an  $A$ -mixed Hodge structure  $(F, W)$  is a collection of non-degenerate bilinear forms

$$Q_k : Gr_k^W \otimes Gr_k^W \rightarrow A(-k)$$

which polarize the pure Hodge structure  $FGr_k^W$  for each index  $k$ .

Given a graded-polarized mixed Hodge structure  $\{(F, W), Q_\bullet\}$  with underlying  $A$ -module  $V_A$  let  $X$  denote the generalized flag variety consisting of all decreasing filtrations  $\tilde{F}$  of  $V_{\mathbb{C}}$  such that  $\dim \tilde{F}^p = \dim F^p$  for each index  $p$ . Let  $\mathcal{M} \subset X$  denote the classifying space of all  $\tilde{F} \in X$  such that

- $(\tilde{F}, W)$  is a mixed Hodge structure with the same graded Hodge numbers as  $(F, W)$ ;
- $Q_\bullet$  is a graded-polarization of  $(\tilde{F}, W)$ .

Let  $GL(V_{\mathbb{C}})^W$  denote the subgroup of  $GL(V_{\mathbb{C}})$  consisting of elements which preserve  $W$ . Define  $G_{\mathbb{C}}$  to be the complex subgroup of  $GL(V_{\mathbb{C}})^W$  consisting of elements which act by isometries of  $Q_\bullet$  on  $Gr^W$ . Let  $G_{\mathbb{R}} = G_{\mathbb{C}} \cap GL(V_{\mathbb{R}})$  and  $G \subset G_{\mathbb{C}}$  be the set consisting of elements of  $G_{\mathbb{C}}$  which act by real isometries of  $Q_\bullet$  on  $Gr^W$ . Then,  $G$  is a real Lie group which acts transitively on  $\mathcal{M}$  by biholomorphisms [32]. The “compact dual”  $\check{\mathcal{M}}$  is defined to be the  $G_{\mathbb{C}}$  orbit of  $F$  in  $X$ . In general,  $\check{\mathcal{M}}$  is not compact due to the fact that  $G_{\mathbb{C}}$  contains the complex subgroup  $G_{-1}$  consisting of element of  $GL(V_{\mathbb{C}})^W$  which act trivially on  $Gr^W$ . Since  $G$  also contains  $G_{-1}$ , viewed as a real Lie group, it follows that  $G_{\mathbb{C}}$  is not the complexification of  $G$  unless  $G_{-1} = 1$ .

See [32] and the references therein for further details on classifying spaces of mixed Hodge structures. In general, variations of mixed Hodge structure do not have good asymptotic behavior in the absence of the existence of a graded-polarization.

**Period Maps.** The axioms of an admissible variation of mixed Hodge structure are given in [39, 24]. As described in [32], given a variation of (graded-polarized) mixed Hodge structure  $\mathcal{V} \rightarrow S$  we let

$$\varphi : S \rightarrow \Gamma \backslash \mathcal{M} \quad (45)$$

denote the corresponding period map, determined by a choice of point  $s_0 \in S$  and  $\Gamma$  is the monodromy group of  $\mathcal{V}$ .

For the remainder of this section, we fix a polydisk  $\Delta^r$  with local coordinates  $(s_1, \dots, s_r)$  and let  $\Delta^{*r}$  denote the complement of  $s_1 \cdots s_r = 0$ . Let  $(z_1, \dots, z_r)$  denote the standard Euclidean coordinates on  $\mathbb{C}^r$  and  $\mathfrak{h}^r$  be the product of upper half-planes defined by  $\text{Im}(z_1), \dots, \text{Im}(z_r) > 0$ . Let  $\mathfrak{h}^r \rightarrow \Delta^{*r}$  be the universal covering defined by  $s_j = e^{2\pi i z_j}$ .

An admissible period map [39, 24]  $\varphi : \Delta^{*r} \rightarrow \Gamma \backslash \mathcal{M}$  with unipotent monodromy then has a local normal form [32]

$$F(z_1, \dots, z_r) = e^{\sum_j z_j N_j} e^{\Gamma(s)} F_\infty \quad (46)$$

where  $T_j = e^{N_j}$ ,  $F_\infty$  is the limit Hodge filtration and  $\Gamma(s)$  is a holomorphic function on  $\Delta^r$  which vanishes at the origin and takes values in a vector space complement to the stabilizer of  $F_\infty$  in  $\text{Lie}(G_{\mathbb{C}})$ . See [32] for details.

The local normal form (46) for variations of pure Hodge structure appears in [8], and works in the setting of admissible  $\mathbb{R}$ -VMHS with unipotent monodromy. Indeed, the special case  $\Gamma(s) = 0$  is an admissible nilpotent orbit (or an infinitesimal mixed Hodge module).

Passage from the period map  $\varphi : \Delta^{*r} \rightarrow \Gamma \backslash \mathcal{M}$  to the local normal form (46) may involve replacing  $\Delta^{*r}$  by a polydisk of smaller radius. Since we are only interested in the asymptotic behavior of the period map in our discussions involving the local normal form, we generally omit this step. However, when it becomes necessary to explicitly restrict to a smaller disk we write

$$\Delta_a^r = \{s \in \Delta^r \mid |s_1|, \dots, |s_r| < a\}. \quad (47)$$

*Remark 48.* Variations of pure, polarized Hodge structure with unipotent monodromy are admissible by the results of Schmid [35] and Cattani, Kaplan and Schmid [9]. If  $\mathcal{V}$  is a variation of pure Hodge structure, we write  $\check{\mathcal{D}}$  and  $\mathcal{D}$  in place of  $\check{\mathcal{M}}$  and  $\mathcal{M}$ .

**2.1. Nilpotent Orbits.** The nilpotent orbit attached to an admissible variation  $\mathcal{V} \rightarrow \Delta^{*r}$  is the map  $\theta(z) = \exp(\sum_j z_j N_j) F_\infty$  from  $\mathbb{C}^r \rightarrow \check{\mathcal{M}}$  obtained by setting  $\Gamma = 0$  in (46). In language of [24], the data  $(N_1, \dots, N_r; F, W)$  determines an infinitesimal mixed Hodge module (IMHM). We shall often use the alternative term admissible nilpotent orbit instead of IMHM. In the case where  $\mathcal{M}$  is a classifying space of pure Hodge structures, an IMHM is just a nilpotent orbit in the sense of Schmid [35]. If  $\mathcal{V} \rightarrow \Delta^{*r}$  is an admissible variation of mixed Hodge structure, we also use the notation  $\mathcal{V}_{\text{nilp}}$  to denote the associated nilpotent orbit.

**Definition 49.** Let  $\mathcal{D}$  be a classifying space of pure Hodge structures of weight  $k$  polarized by  $Q$  and  $\mathfrak{g}_{\mathbb{R}}$  denote the Lie algebra consisting of infinitesimal, real automorphisms of  $Q$ . Then, an  $r$ -variable nilpotent orbit  $\theta$  with values in  $\mathcal{D}$  is a map  $\theta : \mathbb{C}^r \rightarrow \check{\mathcal{D}}$  of the form

$$\theta(z) = \exp\left(\sum_j z_j N_j\right) F \quad (50)$$

where  $N_1, \dots, N_r \in \mathfrak{g}_{\mathbb{R}}$  are commuting nilpotent endomorphisms and  $F \in \check{\mathcal{D}}$  such that

- (i)  $N_j(F^p) \subseteq F^{p-1}$ ;
- (ii) There exists a constant  $a$  such that if  $\text{Im}(z_1), \dots, \text{Im}(z_r) > a$  then  $\theta(z) \in \mathcal{D}$ .

In particular, since  $\mathcal{D}$  encodes the weight of the Hodge structures it parametrizes, the data of a nilpotent orbit with values in  $\mathcal{D}$  reduces to  $(N_1, \dots, N_r; F)$ . If the data of  $\mathcal{D}$  is defined over a subfield  $K$  of  $\mathbb{R}$ , one can define the notion of a  $K$ -nilpotent orbit by requiring that  $N_1, \dots, N_r \in \mathfrak{g}_K = \text{Lie}(\text{Aut}_K(Q))$ .

**Lemma 51.** *If  $(N_1, \dots, N_r; F)$  generate a  $K$ -nilpotent orbit of pure Hodge structure then so does*

$$(t_1 N_1, \dots, t_r N_r; F)$$

for any choice of positive scalars  $t_1, \dots, t_r \in K_+$ .

*Proof.* Conditions (i) and (ii) remain true upon replacing  $N_j$  by  $t_j N_j$ . □

To continue, we recall that if  $N$  is a nilpotent endomorphism of a finite dimensional vector space  $V$  defined over a field of characteristic zero then there exists a unique, increasing filtration  $W = W(N)$  of  $V$  such that [cf. [9]]

- (a)  $N(W_j) \subseteq W_{j-2}$ ;
- (b)  $N^\ell : Gr_\ell^W \rightarrow Gr_{-\ell}^W$  is an isomorphism;

for all  $j$  and  $\ell$ .

**Theorem 52.** (See [9]) *Let  $\theta$  be a nilpotent orbit of pure Hodge structure of weight  $k$  and*

$$\mathcal{C} = \left\{ \sum_j a_j N_j \mid a_1, \dots, a_r > 0 \right\}$$

*Then, the map  $N \mapsto W(N)$  is constant on  $\mathcal{C}$ . Pick  $N \in \mathcal{C}$  and define  $W(N)[-k]_j = W_{j-k}(N)$ . Then,*

$$(F, W(N)[-k]) \quad (53)$$

*is a mixed Hodge structure with respect to which each  $N_j$  is a  $(-1, -1)$ -morphism.*

The pair (53) is called the limit mixed Hodge structure of  $\theta$ . Analysis of the possible nilpotent orbits which can arise can then be reduced to the study of nilpotent orbits with limit mixed Hodge structure which are split over  $\mathbb{R}$  by the following results of Cattani, Kaplan and Schmid [9].

**Theorem 54.** [9] *If  $\theta(z_1, \dots, z_r)$  is a nilpotent orbit of pure Hodge structure such that (53) is split over  $\mathbb{R}$  then  $\theta(z)$  takes values in the appropriate classifying space of pure Hodge structure as soon as  $\text{Im}(z_1), \dots, \text{Im}(z_r) > 0$ .*

**Theorem 55.** [9] If  $\theta(z_1, \dots, z_r)$  is a nilpotent orbit of pure Hodge structure and  $(\tilde{F}, W) = (e^{-i\delta}F, W)$  is the  $\delta$ -splitting of the limit mixed Hodge structure (53) then  $\tilde{\theta}(z_1, \dots, z_r) = e^{\sum_j z_j N_j} \tilde{F}$  is a nilpotent orbit of pure Hodge structure.

The  $SL_2$ -orbit theorem of [35, 9] involves the  $sl_2$  or canonical splitting

$$(F, W) \mapsto (e^{-\xi}F, W) \quad (56)$$

where  $\xi$  is given by universal Lie polynomials in the Hodge components of  $\delta$ . In particular, the mixed Hodge structures  $(F, W)$ ,  $(e^{-i\delta}F, W)$  and  $(e^{-\xi}F, W)$  all induce the same pure Hodge structures on  $Gr^W$  since  $\delta, \xi \in \Lambda_{(F, W)}^{-1, -1}$ .

**Definition 57.** Let  $\theta(z) = e^{\sum_j z_j N_j} F$  be a nilpotent orbit with values in  $\mathcal{D}$  and limit mixed Hodge structure  $(F, W)$ . Let  $(\tilde{F}, W)$  be the Deligne  $\delta$ -splitting of  $(F, W)$ . Then,  $\tilde{\theta}(z) = e^{\sum_j z_j N_j} \tilde{F}$  is called the  $\delta$ -splitting of  $\theta$ . Likewise, if  $(\hat{F}, W)$  is the  $sl_2$ -splitting of  $(F, W)$  then  $\hat{\theta}(z) = e^{\sum_j z_j N_j} \hat{F}$  is called the  $sl_2$ -splitting of  $\theta$ .

*Remark 58.* Theorem (55) holds mutatis mutandis for  $\hat{\theta}$ . Since the property of being a  $K$ -nilpotent orbit does not involve the limit Hodge filtration, passage to the  $\delta$  or  $sl_2$ -splitting does not effect the property of being a  $K$ -nilpotent orbit.

For later use, we record that if  $(F, W)$  is a biextension mixed Hodge structure then

$$W_{-2}gl(V) = gl(V)_{(F, W)}^{-1, -1} \quad (59)$$

for every point  $F \in \mathcal{M}$ . Moreover, in the graded-polarized case,

$$\eta \in Z(\mathfrak{g}_{\mathbb{C}}) \quad (60)$$

i.e.  $\eta$  commutes with every element of  $\mathfrak{g}_{\mathbb{C}}$ . Indeed, any element of  $\mathfrak{g}_{\mathbb{C}}$  must act trivially on  $Gr_0^W$  and  $Gr_{-2}^W$  and  $\eta$  maps  $Gr_0^W$  to  $Gr_{-2}^W$  and annihilates  $W_{-1}(V)$ .

**2.2. Heights of Nilpotent Orbits.** Let  $V$  be a finite dimensional vector space over a field of characteristic zero equipped with a nilpotent endomorphism  $N$  and increasing filtration  $W$  such that  $N(W_k) \subseteq W_k$  for each index  $k$ . Then, there exists at most one increasing filtration  $M = M(N, W)$ , called the relative weight filtration of  $W$  with respect to  $N$  (see [39]), such that

- (a)  $N(M_k) \subseteq M_{k-2}$  for all  $k$ ;
- (b) For any  $k$  and  $\ell$ , the induced map  $N^\ell : Gr_{\ell+k}^M Gr_k^W \rightarrow Gr_{k-\ell}^M Gr_k^W$  is an isomorphism.

**Definition 61.** [39] The data  $(N, F, W)$  defines an admissible nilpotent orbit  $\theta(z) = (e^{zN}F, W)$  with values in the classifying space  $\mathcal{M}$  of graded-polarized mixed Hodge structure provided that

- $N$  is nilpotent, preserves  $W$  and acts by infinitesimal isometries on  $Gr^W$ ;
- $M = M(N, W)$  exists;
- $F \in \check{\mathcal{M}}$  and  $N(F^p) \subseteq F^{p-1}$ ;
- $e^{zN}F$  induces nilpotent orbits of pure, polarized Hodge structure on  $Gr^W$ .

**Theorem 62.** [39] If  $(F, N, W)$  defines an admissible nilpotent orbit and  $M = M(N, W)$  then  $(F, M)$  is a mixed Hodge structure for which  $N$  is a  $(-1, -1)$ -morphism.

As in the pure case,  $(F, M)$  is called the limit mixed Hodge structure of the nilpotent orbit generated by  $(N, F, W)$ . An admissible nilpotent orbit  $\theta$  induces nilpotent orbits of pure Hodge structure on  $Gr^W$  such that the limit mixed Hodge structure of  $\theta$  induces the limit mixed Hodge structure of the graded orbits. The analogs of Theorems (54) and (55) hold mutatis mutandis in the mixed case.

As in Definition (57), one defines the  $\delta$ -splitting and the  $\mathrm{sl}_2$ -splitting of an admissible nilpotent orbit  $\theta$  by replacing by replacing the limit mixed Hodge structure of  $\theta$  with the corresponding Deligne or  $\mathrm{sl}_2$ -splitting.

**Lemma 63** (Deligne [13, 4]). *Let  $(N, F, W)$  generate an admissible nilpotent orbit  $\theta$  with  $\delta$ -splitting generated by  $(N, \tilde{F}, W)$ . Define,*

$$Y(N, F, W) = \mathrm{Ad}(e^{-iN})Y_{(e^{iN}\tilde{F}, W)}, \quad \tilde{Y}(N, F, W) = Y_{(\tilde{F}, M)} \quad (64)$$

Then, writing  $Y = Y(N, F, W)$  and  $\tilde{Y} = \tilde{Y}(N, F, W)$ ,

- (i)  $Y = \tilde{Y}$ ;
- (ii)  $Y$  preserves  $\tilde{F}$ ;
- (ii)  $Y$  commutes with  $\tilde{Y}$  ( $\implies Y$  preserves  $M$ );
- (iii) If  $N = \sum_{j \geq 0} N_{-j}$  relative to  $\mathrm{ad} Y$  then  $N_0$  and  $H = \tilde{Y} - Y$  form an  $\mathrm{sl}_2$ -pair, with associated  $\mathrm{sl}_2$ -triple  $(N_0, H, N_0^+)$ .
- (iv)  $[N - N_0, N_0^+] = 0$ .

*Remark 65.*

- In the language of Deligne systems,  $Y$  is the grading attached to  $N$  and  $\tilde{Y}$  (cf. [4]).
- For  $j > 0$  it follows that  $N_{-j}$  is highest weight  $j - 2$  for  $(N_0, H, N_0^+)$ . This forces  $N_{-1} = 0$  and  $[N_0, N_{-2}] = 0$ .
- In the case where  $(e^{zN}\tilde{F}, W)$  is a biextension mixed Hodge structure then  $N = N_0 + N_{-2}$ .

**Theorem 66** (Theorem (5.19) [31]). *For an admissible biextension variation  $\mathcal{V} \rightarrow \Delta^*$  with nilpotent orbit generated by  $(N, F, W)$ ,*

$$\mu(\mathcal{V}) = N_{-2}\eta \quad (67)$$

where  $N = N_0 + N_{-2}$  defined as in Lemma (63).

*Proof.* Theorem (5.19) of [31] asserts that any grading  $Y$  of  $W$  such that  $[Y, N] \in W_{-2}\mathrm{gl}(V)$  can be used to compute  $\mu(\mathcal{V})$  using (67).  $\square$

**Definition 68.** Let  $(N, F, W)$  generate an admissible nilpotent orbit of biextension type with Deligne splitting  $\delta$  of the limit mixed Hodge structure and  $Y = Y(N, F, W)$  defined as in (64). Let  $\delta = \sum_{k \leq 0} \delta_k$  with  $[Y, \delta_k] = k\delta_k$ . Then, the limit height  $ht(N, F, W)$  of the nilpotent orbit  $(e^{zN}F, W)$  is given by the formula

$$2\pi\delta_{-2} = ht(N, F, W)\eta \quad (69)$$

*Remark 70.* A biextension mixed Hodge structure  $(F, W)$  on  $V$  determines an admissible nilpotent orbit by setting  $N = 0$ . In this case  $M = W$ ,  $\delta$  is the Deligne splitting of  $(F, W)$  and  $Y(N, F, W)$  is a grading of  $W$ . Since  $\delta \in W_{-2}\mathrm{gl}(V)$  it then follows from the short length of  $W$  that  $\delta = \delta_{-2}$  relative to  $Y(N, F, W)$  and hence  $ht(N, F, W) = h(F, W)$ .

### 3. REGULARITY RESULTS

The following result justifies the name limit height (Theorem (74)): Let  $\mathcal{V} \rightarrow \Delta^*$  be an admissible biextension VMHS with nilpotent orbit  $(e^{zN}F_\infty, W)$  and asymptotic height  $h(\mathcal{V}) \cong -\mu \log |s|$ . Then,

$$\bar{h}(s) = h(\mathcal{V}) + \mu \log |s|$$

has a continuous extension to  $\Delta$  with  $\bar{h}(0) = ht(N, F_\infty, W)$ .

**SL<sub>2</sub>-orbits.** In this section we review the theory of SL<sub>2</sub> orbits for nilpotent orbits of biextension type following [31]. Let  $\mathcal{M}$  be the ambient classifying space with associated Lie group  $G$  and  $\tilde{G}$  be the subgroup of  $G$  which acts as real transformations on  $W_k/W_{k-2}$  for all  $k$ .

Let  $(e^{zN}F, W)$  be an admissible nilpotent orbit of biextension type with limit mixed Hodge structure  $(F, M)$ , let  $(\tilde{F}, M) = (e^{-i\delta}F, M)$  denote the Deligne splitting of  $(F, M)$  and  $\tilde{Y} = Y(N, F, W)$ ,  $Y = Y(N, F, W)$ , and  $N = N_0 + N_{-2}$ ,  $N_0^+$  etc. as in Lemma (63).

To continue, we recall the SL<sub>2</sub>-orbit Theorem of [31]:

**Theorem 71** (Theorem 4.2 [31]). *Let  $(e^{zN}F, W)$  be an admissible nilpotent orbit of biextension type. Then, there exists*

$$\chi \in \text{Lie}(\tilde{G}) \cap \ker(N) \cap \Lambda_{(\tilde{F}, M)}^{-1, -1}$$

and distinguished real analytic function  $g : (a, \infty) \rightarrow \tilde{G}$  such that

- (a)  $e^{iyN}.F = g(y)e^{iyN}.\tilde{F}$ ;
- (b)  $g(y)$  and  $g^{-1}(y)$  have convergent series expansions about  $\infty$  of the form

$$\begin{aligned} g(y) &= e^\chi (1 + g_1 y^{-1} + g_2 y^{-2} + \dots) \\ g^{-1}(y) &= (1 + f_1 y^{-1} + f_2 y^{-2} + \dots) e^{-\chi} \end{aligned}$$

with  $g_k, f_k \in \ker(\text{ad } N_0)^{k+1} \cap \ker(\text{ad } N_{-2})$ ;

- (c)  $\delta, \chi$  and the coefficients  $g_k$  are related by the formula

$$e^{i\delta} = e^\chi \left( 1 + \sum_{k \geq 0} \frac{1}{k!} (-i)^k (\text{ad } N_0)^k g_k \right)$$

Moreover,  $\chi$  can be expressed as a universal Lie polynomial over  $\mathbb{Q}(\sqrt{-1})$  in the Hodge components  $\delta^{r,s}$  of  $\delta$  with respect to  $(\tilde{F}, W)$ . Likewise, the coefficients  $g_k$  and  $f_k$  can be expressed as universal, non-commuting polynomials over  $\mathbb{Q}(\sqrt{-1})$  in  $\delta^{r,s}$  and  $\text{ad } N_0^+$ .

**Corollary 72** (Corollary (4.3) [31]). *Let  $\mathcal{V} \rightarrow \Delta^*$  be an admissible biextension variation, with period map  $F(z) : \mathfrak{h} \rightarrow \mathcal{M}$  and nilpotent orbit  $e^{zN}F$ . Let  $F_0 = e^{iN_0}\tilde{F}$ . Then, (cf. Theorem (71)), there exists a distinguished, real-analytic function  $\gamma(z)$  with values in  $\text{Lie}(\tilde{G})$  such that, for  $\text{Im}(z)$  sufficiently large,*

- (i)  $F(z) = e^{xN}g(y)e^{iyN_{-2}}y^{-H/2}e^{\gamma(z)}.F_0$ ;
- (ii)  $|\gamma(z)| = O(\text{Im}(z)^\beta e^{-2\pi\text{Im}(z)})$  as  $y \rightarrow \infty$  and  $x$  restricted to a finite subinterval of  $\mathbb{R}$ , for some constant  $\beta \in \mathbb{R}$ .

**Punctured Disk.** Let  $\mathcal{V} \rightarrow \Delta^*$  be an admissible biextension with nilpotent orbit  $(e^{zN}F, W)$  and define  $\delta, Y, \tilde{Y}$  etc. as in the previous subsection. Observe that since  $\tilde{Y}$  and  $Y$  commute, we can write

$$\delta = \sum_{a,b} \delta_{a,b}, \quad [\tilde{Y}, \delta_{a,b}] = a\delta_{a,b}, \quad [Y, \delta_{a,b}] = b\delta_{a,b}$$

Define  $\chi = \sum_{a,b} \chi_{a,b}$  similarly. Let

$$\delta(j) = \sum_{a-b=j} \delta_{a,b}, \quad \chi(j) = \sum_{a-b=j} \chi_{a,b}$$

and note that  $\delta(j)$  and  $\chi(j)$  are the respective projections of  $\delta$  and  $\chi$  onto the  $j$ -eigenspace of  $H$ . In particular, since  $N = N_0 + N_{-2}$  and  $N_{-2}$  is central in  $\mathfrak{g}_{\mathbb{C}}$  it follows that  $[N_0, \delta(j)] = 0$  and  $[N_0, \chi(j)] = 0$ . Consequently,  $\delta(j)$  and  $\chi(j)$  vanish for  $j > 0$ .

**Lemma 73.**  $g^\dagger(y) = \text{Ad}(y^{H/2})g(y)$  is given by a convergent power series in powers of  $y^{-1/2}$  with

$$\lim_{y \rightarrow \infty} g^\dagger(y) = e^{i\delta_{-2}}$$

where  $\delta = \delta_0 + \delta_{-1} + \delta_{-2}$  relative to  $\text{ad } Y$ .

*Proof.* To compute the limiting value, it is sufficient to show that

- (a)  $\lim_{y \rightarrow \infty} \text{Ad}(y^{H/2})(e^{-\chi}g(y)) = 1$ ;
- (b)  $\lim_{y \rightarrow \infty} \text{Ad}(y^{H/2})e^\chi = e^{i\delta_{-2}}$ ;

The intermediate computations will show that  $g^\sharp(y)$  has a power series expansion of the stated form.

To verify (a), recall that

$$g(y) = e^\chi(1 + g_1 y^{-1} + g_2 y^{-2} + \dots)$$

with  $g_k \in \ker(\text{ad}(N_0)^{k+1})$ . Let  $g(y)(j)$  denote the projection of  $e^{-\chi}g(y)$  to the  $j$ -eigenspace of  $\text{ad}(H)$ . (Note the slight conflict with the notation used above for  $\delta$  and  $\chi$ .) Define

$$g_-(y) = \sum_{j < 0} g(y)(j), \quad g_0(y) = g(y)(0), \quad g_+(y) = \sum_{j > 0} g(y)(j)$$

Then,

$$e^{-\chi}g(y) = g_-(y) + g_0(y) + g_+(y).$$

By construction,  $\text{Ad}(y^{H/2})g_-(y)$  is a convergent power series in  $y^{-1/2}$  with constant term zero. Likewise,  $\text{Ad}(y^{H/2})g_0(y) = g(y)(0)$  is a convergent series in  $y^{-1/2}$  with constant term 1.

To continue, observe that  $g_k(j) = 0$  unless  $k \geq j$  since  $\text{ad}(N_0)^{k+1}g_k = 0$ . Consequently, for  $j > 0$  we have

$$g(y)(j) = \sum_{k \geq j} g_k(j) y^{-k} = y^{-j} \sum_{k \geq j} g_k(j) y^{j-k}$$

As such, for  $j > 0$ ,  $\text{Ad}(y^{H/2})g(y)(j)$  is a convergent power series in  $y^{-1/2}$  with constant term zero. Combining this with the previously computations, this shows that  $g^\sharp(y)$  has a convergent series expansion in  $y^{-1/2}$  with constant term 1. This establishes (a).

To verify (b), we note that

$$e^{iyN} e^{i\delta} \tilde{F} = e^\chi e^{-\chi} g(y) e^{iyN} \tilde{F}.$$

Since  $N_{-2}$  is central in  $\mathfrak{g}_\mathbb{C}$ , it follows that

$$e^{iyN_0} e^{i\delta} \tilde{F} = e^\chi e^{-\chi} g(y) e^{iyN_0} \tilde{F}.$$

Applying  $y^{H/2}$  to each side of this equation, and using the fact that  $H$  preserves  $\tilde{F}$  it follows that

$$e^{iN_0} e^{i \sum_{j \leq 0} y^{j/2} \delta(j)} \tilde{F} = e^{\sum_{j \leq 0} y^{j/2} \chi(j)} \text{Ad}(y^{H/2})(e^{-\chi}g(y)) e^{iN_0} \tilde{F}$$

Using part (a), and taking the limit at  $y \rightarrow \infty$  yields

$$e^{iN_0} e^{i\delta(0)} \tilde{F} = e^{\chi(0)} e^{iN_0} \tilde{F}$$

In particular, since  $[N_0, \chi] = 0$  it follows that  $[N_0, \chi(j)] = 0$  for each  $j$ . Applying  $e^{-iN_0}$  to the previous equation therefore implies

$$e^{i\delta(0)} \tilde{F} = e^{\chi(0)} \tilde{F}$$

Finally,  $\delta(0)$  and  $\chi(0)$  take values in  $\Lambda_{(\tilde{F}, M)}^{-1, -1}$  and the map  $\lambda \in \Lambda_{(\tilde{F}, M)}^{-1, -1} \rightarrow e^\lambda \tilde{F}$  is injective. Thus,  $i\delta(0) = \chi(0)$ .

To finish the proof of part (b), observe that  $\delta$  is real and  $\chi$  is required to act by real transformations on  $W_k/W_{k-2}$ . Therefore  $i\delta(0) = \chi(0)$  implies that  $\chi(0) = \chi_{-2, -2} = i\delta_{-2, -2}$ . Indeed,

$$i\delta(0) = \chi(0) \iff i\delta_{p,p} = \chi_{p,p}$$

for each  $p$ . In order for  $\chi$  to act by real transformations on  $W_k/W_{k-2}$  we must have  $p = -2$ . Since  $W_{-2}\text{Lie}(\mathbb{G}_\mathbb{C})$  is central,

$$[\tilde{Y}, \delta_{-2}] = [H + Y, \delta_{-2}] = [Y, \delta_{-2}] = -2\delta_{-2}$$

Consequently,  $\chi(0) = i\delta_{-2,-2} = i\delta_{-2}$ . Using the fact that  $\chi(j) = 0$  for  $j > 0$ , the computation of the limit in (b) then follows directly.  $\square$

**Theorem 74.** *Let  $\mathcal{V} \rightarrow \Delta^*$  be an admissible biextension with unipotent monodromy  $T = e^N$  and limit mixed Hodge structure  $(F, M)$ . Then*

$$\lim_{s \rightarrow 0} h(s) = \lim_{s \rightarrow 0} h(\mathcal{V}) + \mu \log |s| = ht(N, F, W).$$

*Proof.* Let  $(\tilde{F}, M) = (e^{-i\delta}F, M)$  be Deligne's  $\delta$ -splitting of the limit mixed Hodge structure  $(F, M)$  of  $\mathcal{V}$ . Let  $F(z)$  be the lifting of the period map of  $\mathcal{V}$  to the upper half-plane. By Corollary (72), we have

$$\delta_{(F(z), W)} = \delta_{(e^{xN}g(y)e^{iyN-2}y^{-H/2}e^{\gamma(z)}.F_0, W)}$$

where  $F_0 = e^{iN_0}\tilde{F}$ , and we use  $g.F_0$  instead of  $gF$  to break up long computations. Moreover,  $|\gamma(z)| = O(\text{Im}(z)^\beta e^{-2\pi\text{Im}(z)})$  as  $y \rightarrow \infty$  and  $x$  restricted to a finite subinterval of  $\mathbb{R}$ . Since  $xN$  is real, it follows from that

$$\begin{aligned} \delta_{(F(z), W)} &= \text{Ad}(e^{xN})\delta_{(g(y)e^{iyN-2}y^{-H/2}e^{\gamma(z)}.F_0, W)} \\ &= \delta_{(g(y)e^{iyN-2}y^{-H/2}e^{\gamma(z)}.F_0, W)}. \end{aligned}$$

Likewise, since  $N_{-2} \in W_{-2}\mathfrak{g}_{\mathbb{C}}$  is central, it follows (using (39) and (41)) that

$$\begin{aligned} \delta_{(F(z), W)} &= \delta_{(g(y)e^{iyN-2}y^{-H/2}e^{\gamma(z)}.F_0, W)} \\ &= \delta_{(g(y)y^{-H/2}e^{\gamma(z)}.F_0, W)} + yN_{-2}. \end{aligned}$$

Consequently,

$$\begin{aligned} \delta_{(F(z), W)} - yN_{-2} &= \delta_{(g(y)y^{-H/2}e^{\gamma(z)}.F_0, W)} \\ &= \delta_{(y^{-H/2}y^{H/2}g(y)y^{-H/2}e^{\gamma(z)}.F_0, W)} \\ &= \text{Ad}(y^{-H/2})\delta_{(y^{H/2}g(y)y^{-H/2}e^{\gamma(z)}.F_0, W)} \\ &= \delta_{(y^{H/2}g(y)y^{-H/2}e^{\gamma(z)}.F_0, W)} \end{aligned}$$

since  $W_{-2}gl(V) \subset Z(\mathfrak{g}_{\mathbb{C}})$  and hence  $\text{Ad}(y^{-H/2})$  acts trivially on  $W_{-2}gl(V)$ . Therefore,

$$\lim_{\text{Im}(z) \rightarrow \infty} \delta_{(F(z), W)} - yN_{-2} = \delta_{(e^{i\delta-2}e^{iN_0}.\tilde{F}, W)} \quad (75)$$

for  $\text{Re}(z)$  restricted to an interval of finite length.

To finish the proof, we note that  $(e^{iN_0}.\tilde{F}, W)$  is a mixed Hodge structure which is split over  $\mathbb{R}$  with  $Y_{(e^{iN_0}.\tilde{F}, W)} = Y$ . Consequently,

$$\delta_{(e^{i\delta-2}e^{iN_0}.\tilde{F}, W)} = \delta_{-2} + \delta_{(e^{iN_0}.\tilde{F}, W)} = \delta_{-2}.$$

Therefore, using equations (69), (67) and (39), equation (75) becomes

$$\lim_{\text{Im}(z) \rightarrow \infty} h(F(z), W) + \mu \log |s| = ht(N, F, W).$$

$\square$

**Smooth Divisor.** Consider now the case where  $\mathcal{V}$  is a biextension variation over the complement of  $s_1 = 0$  in a polydisk  $\Delta^r$  with coordinates  $(s_1, \dots, s_r)$ . Let

$$F(z_1, s_2, \dots, s_r) = e^{z_1 N_1} e^{\Gamma(s)}.F_\infty \quad (76)$$

denote the local normal form of the period map of  $\mathcal{V}$  after lifting to  $U \times \Delta^{r-1}$  where  $U$  is the upper half-plane with coordinate  $z_1$  and  $s_1 = e^{2\pi i z_1}$ . To simplify notation, we shall write  $N$  in place of  $N_1$  and  $z = x + iy$  in place of  $z_1$  where convenient.



To continue, we recall [22] that

$$[N_1, \Gamma(0, s_2, \dots, s_r)] = 0$$

Accordingly, we let

$$\Gamma_1 = \Gamma(0, s_2, \dots, s_r), \quad e^{\Gamma(s)} = e^{\Gamma_1} e^{\Gamma_1}$$

and note that  $\Gamma^1(0, s_2, \dots, s_r) = 0$  by construction, and hence  $s_1 | \Gamma^1$ .

Let  $(\tilde{F}_\infty, M) = (e^{-i\delta} \cdot F_\infty, M)$  be the Deligne splitting of the limit mixed Hodge structure of  $\mathcal{V}$ . Define  $N = N_0 + N_{-2}$  and  $H = \tilde{Y} - Y$  etc. as in Theorem (71) for the nilpotent orbit  $(e^{zN} \cdot F_\infty, W)$ . Then, since  $N_{-2}$  is central in  $\mathfrak{g}_\mathbb{C}$  and  $[N, \Gamma_1] = 0$  it follows that  $[N_0, \Gamma_1] = 0$ . This forces

$$\Gamma_1 = \sum_{k \leq 0} \Gamma_{1,k}, \quad [H, \Gamma_{1,k}] = k \Gamma_{1,k}.$$

Let  $\Gamma^1 = \sum_k \Gamma_k^1$  where  $[H, \Gamma_k^1] = k \Gamma_k^1$ .

**Theorem 77.** *Let  $\mathcal{V}$  be a biextension variation with unipotent monodromy defined on the complement of the divisor  $s_1 = 0$  in  $\Delta^r$ . Then,*

$$\bar{h}(\mathcal{V}) = h(\mathcal{V}) + \mu \log |s_1|$$

*extends continuously to  $\Delta^r$ .*

*Proof.* By the above remarks, we can write

$$\begin{aligned} F(z, s_2, \dots, s_r) &= e^{xN} e^{iyN} e^{\Gamma(s)} e^{-iyN} e^{iyN} \cdot F_\infty \\ &= e^{xN} e^{iyN} e^{\Gamma^1(s)} e^{\Gamma_1(s)} e^{-iyN} e^{iyN} \cdot F_\infty \\ &= e^{xN} e^{iyN} e^{\Gamma^1(s)} e^{-iyN} e^{\Gamma_1(s)} e^{iyN} \cdot F_\infty \\ &= e^{xN} e^{iyN_0} e^{iyN_{-2}} e^{\Gamma^1(s)} e^{-iyN_{-2}} e^{-iyN_0} e^{\Gamma_1(s)} e^{iyN} \cdot F_\infty \\ &= e^{xN} e^{iyN_0} e^{\Gamma^1(s)} e^{-iyN_0} e^{\Gamma_1(s)} e^{iyN} \cdot F_\infty. \end{aligned}$$

By theorem (71), we can then write

$$\begin{aligned} e^{iyN} F_\infty &= g(y) e^{iyN} \cdot \tilde{F}_\infty \\ &= g(y) e^{iyN_{-2}} e^{iyN_0} \tilde{F}_\infty \\ &= e^{iyN_{-2}} g(y) e^{iyN_0} \tilde{F}_\infty \\ &= e^{iyN_{-2}} g(y) y^{-H/2} e^{iN_0} \tilde{F}_\infty \\ &= e^{iyN_{-2}} y^{-H/2} g^\dagger(y) e^{iN_0} \tilde{F}_\infty \end{aligned}$$

where  $g^\dagger(y) = \text{Ad}(y^{H/2})g(y)$  as in Lemma (73).

Combining the previous two paragraphs yields

$$\begin{aligned} \delta_{(F(z, s_2, \dots, s_r), W)} &= \delta_{(e^{xN} e^{iyN_0} e^{\Gamma^1(s)} e^{-iyN_0} e^{\Gamma_1(s)} e^{iyN} \cdot F_\infty, W)} \\ &= \delta_{(e^{iyN_0} e^{\Gamma^1(s)} e^{-iyN_0} e^{\Gamma_1(s)} e^{iyN} \cdot F_\infty, W)} \\ &= \delta_{(e^{iyN_0} e^{\Gamma^1(s)} e^{-iyN_0} e^{\Gamma_1(s)} e^{iyN_{-2}} y^{-H/2} g^\dagger(y) e^{iN_0} \tilde{F}_\infty, W)} \\ &= \delta_{(e^{iyN_{-2}} e^{iyN_0} e^{\Gamma^1(s)} e^{-iyN_0} e^{\Gamma_1(s)} y^{-H/2} g^\dagger(y) e^{iN_0} \tilde{F}_\infty, W)} \\ &= \delta_{(e^{iyN_0} e^{\Gamma^1(s)} e^{-iyN_0} e^{\Gamma_1(s)} y^{-H/2} g^\dagger(y) e^{iN_0} \tilde{F}_\infty, W)} + yN_{-2}. \end{aligned}$$

To complete the proof, let  $\Gamma_{\dagger}^1 = \text{Ad}(y^{H/2})\Gamma^1$ , and observe that  $\Gamma_{\dagger}^1$  is a finite sum of functions  $(\log |s_1|)^{k/2} s_1 f$  where  $f$  is a holomorphic function on  $\Delta^r$  with values in  $\mathfrak{q}$ , and hence has a continuous extension to  $\Delta^r$ . Likewise,

$$\Gamma_1^{\dagger} = \text{Ad}(y^{H/2})\Gamma_1$$

is a finite sum of functions of the form  $(\log |s_1|)^{-k/2} f$  for  $k \geq 0$  and  $f$  a holomorphic function on  $\Delta^r$  with values in  $\mathfrak{q}$  which is independent of  $s_1$ , and hence also has a continuous extension to  $\Delta^r$ . Finally,  $g^{\dagger}(y)$  is independent of  $(s_2, \dots, s_r)$  and has a convergent series expansion in powers of  $(\log |s_1|)^{-1/2}$  by Lemma (73).

Combining the above, it follows that

$$\begin{aligned} \delta_{(F(z, s_2, \dots, s_r), W)} - yN_{-2} &= \delta_{(e^{iyN_0} e^{\Gamma_1(s)} e^{-iyN_0} e^{\Gamma_1(s)} y^{-H/2} g^{\dagger}(y) e^{iN_0} \tilde{F}_{\infty}, W)} \\ &= \delta_{(y^{-H/2} e^{iN_0} e^{\Gamma_1^{\dagger}(s)} e^{-iN_0} e^{\Gamma_1^{\dagger}(s)} g^{\dagger}(y) e^{iN_0} \tilde{F}_{\infty}, W)} \\ &= \text{Ad}(y^{-H/2}) \delta_{(e^{iN_0} e^{\Gamma_1^{\dagger}(s)} e^{-iN_0} e^{\Gamma_1^{\dagger}(s)} g^{\dagger}(y) e^{iN_0} \tilde{F}_{\infty}, W)} \\ &= \delta_{(e^{iN_0} e^{\Gamma_1^{\dagger}(s)} e^{-iN_0} e^{\Gamma_1^{\dagger}(s)} g^{\dagger}(y) e^{iN_0} \tilde{F}_{\infty}, W)} \end{aligned}$$

since  $\text{Ad}(y^{-H/2})$  acts trivially on  $W_{-2}\mathfrak{g}_{\mathbb{C}}$ . By the remarks of the previous paragraph

$$e^{iN_0} e^{\Gamma_1^{\dagger}(s)} e^{-iN_0} e^{\Gamma_1^{\dagger}(s)} g^{\dagger}(y) e^{iN_0}$$

has a continuous extension to  $\Delta^r$ , and hence so does  $\delta_{(F(z, s_2, \dots, s_r), W)} - yN_{-2}$ . Using equations (69), (67) and (39), it follows as in the last paragraph of the proof of Theorem (74) that  $\bar{h}(\mathcal{V})$  has a continuous extension to  $\Delta^r$ .  $\square$

*Remark 78.* Since  $\bar{h}(\mathcal{V})$  extends continuously to  $\Delta^r$  its value at any point  $(0, s_2, \dots, s_r)$  can be computed by taking the limit along  $s \rightarrow (s, s_2, \dots, s_r)$ . Since this corresponds to a 1 parameter degeneration, it follows from Theorem (74) that  $\bar{h}(\mathcal{V})(0, s_2, \dots, s_r)$  depends only on the limit mixed Hodge structure of  $\mathcal{V}$  at  $(0, s_2, \dots, s_r)$ .

#### 4. EXISTENCE OF BIEXTENSIONS

In this section we prove the following result, which shows that over  $\Delta^{*r}$  a pair of admissible normal functions can be glued together to give an admissible biextension. For the remainder of this section,  $A$  is either  $\mathbb{Z}$ ,  $\mathbb{Q}$  or  $\mathbb{R}$ , and

- $\Delta^r$  is a polydisk with coordinates  $(s_1, \dots, s_r)$ ;
- $D$  is the divisor defined by  $s_1 \cdots s_r = 0$ ;
- $\mathcal{H}$  is a torsion free, polarized variation of pure  $A$ -Hodge structure of weight  $-1$  on  $\Delta^r - D$  with quasi-unipotent monodromy;
- $\nu$  is an admissible extension of  $A - \text{VMHS}$

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{V}_{\nu} \rightarrow A(0) \rightarrow 0$$

with respect to inclusion of  $\Delta^r - D$  into  $\Delta^r$ ;

- $\omega$  is an admissible extension of  $A - \text{VMHS}$

$$0 \rightarrow A(1) \rightarrow \mathcal{V}_{\omega}^{\vee} \rightarrow \mathcal{H} \rightarrow 0$$

with respect to inclusion of  $\Delta^r - D$  into  $\Delta^r$ .

**Theorem 79.** *The set  $B(\nu, \omega)$  of admissible biextension variations of type  $(\nu, \omega)$  over  $\Delta^r - D \hookrightarrow \Delta^r$  is non-empty.*

We prove Theorem (79) in the case of unipotent monodromy first, and hence assume that all VMHS have unipotent monodromy unless otherwise noted. By formally allowing the monodromy about  $s_j = 0$  to be trivial, we can reduce the proof of Theorem (79) to the case where  $D$  is the divisor  $s_1 \dots s_r = 0$ . The outline of the proof is as follows: The normal functions  $\nu$  and  $\omega$  have local normal forms (46). Likewise, as a consequence of the  $SL_2$ -orbit theorem if  $F : \mathfrak{h} \rightarrow \mathcal{M}$  is the period map of an admissible nilpotent orbit then

$$\lim_{\text{Im}(z) \rightarrow \infty} Y_{(F(z), W)}$$

exists as a grading of  $W$ . Moreover, this grading will preserve the  $I^{p,q}$ 's of the  $sl_2$ -splitting of the limit mixed Hodge structure of  $(F(z), W)$ . This produces a trigrading of the underlying vector space. For  $\nu$  and  $\omega$ , we must then glue these two trigradings together, and use the resulting new vector space as the model for the biextension variation. It remains then to add an integral structure, local monodromy logarithms et. cetera.

A sketch of the proof of Theorem (79) in the quasi-unipotent case is as follows: A quasi-unipotent VMHS can be viewed as a VMHS on the pullback to the unipotent setting which has the extra property that when the  $i$ 'th coordinate is multiplied by a suitable root of unity, the Hodge filtration changes by the appropriate semisimple factor. Gluing  $\nu$  and  $\omega$  together in the quasi-unipotent case therefore reduces to the unipotent case, together with the fact that the semisimple parts of the local monodromy are morphisms of Hodge structure in the limit, and are therefore compatible with the gluing procedure.

**Trigraded Vector Spaces.** The *weight filtration* of a trigraded vector space

$$L = \bigoplus_{p,q,i} L_i^{p,q} \quad (80)$$

is the increasing filtration  $W_k(L) = \bigoplus_{i \leq k, p,q} L_i^{p,q}$ . The subspaces  $L_i = \bigoplus_{p,q} L_i^{p,q}$  define a grading of  $W_\bullet(L)$ . Likewise, we set  $L^{p,q} = \bigoplus_i L_i^{p,q}$  and call

$$L = \bigoplus_{p,q} L^{p,q}$$

is the associated *bigrading* of  $L$ . We define the *relative weight filtration* of  $L$  to be the increasing filtration  $M_k(L) = \bigoplus_{p+q \leq k, i} L_i^{p,q}$ . If  $L$  is a complex vector space with real form  $L_{\mathbb{R}}$  then a trigrading of  $L$  is said to split over  $\mathbb{R}$  if  $\bar{L}_i^{p,q} = L_i^{q,p}$ . In this case, the decreasing filtration  $F^p(L) = \bigoplus_{a \geq p, b} L^{a,b}$  pairs with  $M$  to define an  $\mathbb{R}$ -split mixed Hodge structure  $(F, M)$  which induces pure Hodge structures on  $Gr^W$  and has Deligne bigrading  $I^{p,q} = L^{p,q}$ .

If  $U$  and  $V$  are vector spaces equipped with linear maps  $f : U \rightarrow Z$  and  $g : V \rightarrow Z$  then the fiber product

$$U \times_Z V = \{(u, v) \in U \times V \mid f(u) = g(v)\}$$

is called the *gluing* of  $U$  and  $V$  along  $Z$ . We let  $\pi_U : U \times_Z V \rightarrow U$  and  $\pi_V : U \times_Z V \rightarrow V$  denote the natural projection maps.

Let  $U$  and  $V$  be trigraded vector spaces. Then  $U$  *abuts*  $V$  if there exists an index  $\ell$  such that

- (a)  $U_i = 0$  for  $i > \ell$  and  $V_i = 0$  for  $i < \ell$ ;
- (b) There exists an isomorphism  $Gr_{\ell}^W(U) \cong Gr_{\ell}^W(V)$  for which the composite map

$$\sigma : U_{\ell} \rightarrow Gr_{\ell}^W(U) \cong Gr_{\ell}^W(V) \rightarrow V_{\ell}$$

induces an isomorphism  $U_{\ell}^{p,q} \rightarrow V_{\ell}^{p,q}$  for each bi-index  $(p, q)$ .

If  $U$  abuts  $V$  (at  $\ell$ ) we define  $Gr_\ell^W \subset Gr_\ell^W(U) \times Gr_\ell^W(V)$  to be the subspace consisting of points  $([u], [v])$  which are identified under the isomorphism  $(b)$ . We then define

$$U \star V = U \times_{Gr_\ell^W} V$$

relative to the linear maps  $f : U \rightarrow U_\ell \rightarrow Gr_\ell^W$  and  $g : V \rightarrow V_\ell \rightarrow Gr_\ell^W$ .

Let  $\iota_U$  from  $U \rightarrow U \star V$  denote the linear map which sends  $u \in U_i$  to  $(u, 0) \in U \star V$  for  $i < \ell$  and maps  $u \in U_\ell$  to  $(u, \sigma(u)) \in U \star V$ . Let  $\iota_V$  from  $V \rightarrow U \star V$  be the map that send  $v \in V_i$  to  $(0, v) \in U \star V$  for  $i > \ell$  and maps  $v \in V_\ell$  to  $(\sigma^{-1}(v), v) \in U \star V$ . Accordingly, we can introduce a trigrading on  $U \star V$  by setting

$$(U \star V)_i^{p,q} = \begin{cases} \iota_U(U_i^{p,q}), & i \leq \ell \\ \iota_V(V_i^{p,q}), & i > \ell \end{cases} \quad (81)$$

If the trigradings of  $U$  and  $V$  are split over  $\mathbb{R}$  and the isomorphism  $Gr_\ell^W(U) \rightarrow Gr_\ell^W(V)$  intertwines complex conjugation then the trigrading (81) is split over  $\mathbb{R}$ .

**Endomorphisms.** Let  $U$  and  $V$  be trigraded vector spaces which abut (at  $\ell$ ). Let  $\alpha_U$  and  $\alpha_V$  be endomorphisms of  $U$  and  $V$  respectively which preserve the corresponding weight filtrations  $W(U)$  and  $W(V)$ . Assume that  $\alpha_U$  and  $\alpha_V$  induce the same map on  $Gr_\ell^W$ . For  $\eta \in (U \star V)_i$  define

$$(\alpha_U \star \alpha_V)(\eta) = \begin{cases} \iota_U \circ \alpha_U \circ \pi_U(\eta), & i \leq \ell \\ \iota_V \circ \alpha_V \circ \pi_V(\eta), & i > \ell. \end{cases} \quad (82)$$

Then,  $\alpha = \alpha_U \star \alpha_V$  is an endomorphism of  $U \star V$  which preserves  $W(U \star V)$  such that

$$\pi_U \circ \alpha \circ \iota_U = \alpha_U, \quad \pi_V \circ \alpha \circ \iota_V = \alpha_V.$$

.

**Lemma 83.** *The construction (82) is compatible with the bigrading (81), i.e. if  $\alpha_U(U^{a,b}) \subseteq U^{p+a,q+b}$  and  $\alpha_V(V^{a,b}) \subseteq V^{p+a,q+b}$  then*

$$\alpha(U \star V)^{a,b} \subseteq (U \star V)^{p+a,q+b} \quad (84)$$

*Proof.* This follows from that fact that  $\pi_U \circ \iota_U$  is the identity on  $U$ ,  $\pi_V \circ \iota_V$  is the identity on  $V$  and both the maps  $\pi_U, \pi_V, \iota_U, \iota_V$  all preserve the relevant bigradings by construction.  $\square$

**Limiting Splitting.** Let  $\psi : \Delta^* \rightarrow \Gamma \backslash \mathcal{M}$  be the period map of an admissible extension of  $\mathbb{R}(0)$  by a variation of pure, polarized  $\mathbb{R}$ -Hodge structure of weight  $-1$ . Let  $V$  denote the underlying vector space of  $\psi$ , let  $W$  denote the weight filtration, and assume the monodromy is unipotent and given by  $T = e^N$ . Let

$$F(z) : \mathfrak{h} \rightarrow \mathcal{M}, \quad F(z+1) = TF(z)$$

be a lifting of  $\psi$  to a map from the upper half-plane  $\mathfrak{h}$  into  $\mathcal{M}$ . Then, by [5, Theorem 3.9], it follows that

$$Y^\dagger = \lim_{\text{Im}(z) \rightarrow \infty} Y_{(F(z), W)} \quad (85)$$

exists, where the limit is taken with  $\text{Re}(z)$  restricted to a finite interval. More precisely,

$$Y^\dagger = \text{Ad}(e^{-iN})Y_{(e^{iN}\hat{F}, W)}$$

where  $(\hat{F}, M) = (e^{-\zeta}F, M)$  is  $\mathfrak{sl}_2$ -splitting of the limit mixed Hodge structure  $(F, M)$  of  $F : \mathfrak{h} \rightarrow \mathcal{M}$ .

**Corollary 86.** *The subspaces*

$$L_k^{p,q} = \{ v \in I_{(\hat{F}, M)}^{p,q} \mid Y^\dagger(v) = kv \}$$

*form a trigrading of  $V$  which is split over  $\mathbb{R}$  such that  $W(L) = W$ ,  $F(L) = \hat{F}$  and  $M(L) = M$  as in (80).*

For future use, we record that in this setting, the element  $\zeta$  appearing (56) in the  $\mathfrak{sl}_2$ -splitting of  $(F, M)$  also preserves the weight filtration  $W$  [4]. The next result is only used in the quasi-unipotent case:

**Proposition 87.** *If  $\gamma$  is a morphism of  $(F, M)$  which preserves  $W$  such that  $[\gamma, N] = 0$  then  $[\gamma, Y^\dagger] = 0$ .*

*Proof.* Since  $\gamma$  is a morphism of  $(F, M)$  it follows that  $\gamma$  commutes with the Hodge components  $\{\delta^{p,q}\}$  of the Deligne splitting of  $(F, M)$ . As  $\zeta$  is given by universal Lie polynomials in the Hodge components of  $\delta$ , it follows that  $\gamma$  commutes with  $\zeta$ . Therefore,  $\gamma$  is a morphism of  $(\hat{F}, M) = (e^{-\zeta}F, M)$ . By [5],

$$Y^\dagger = \text{Ad}(e^{-iN})Y_{(e^{iN}\hat{F}, W)}.$$

As  $\gamma$  commutes with  $N$  and preserves  $W$ , it follows that  $\gamma$  is of type  $(0, 0)$  with respect to  $(e^{iN}\hat{F}, W)$  as well. Thus,  $[\gamma, Y^\dagger] = 0$ .  $\square$

**Limit Mixed Hodge Structure.** We now return to the notation from the beginning of this section. In particular,  $\mathcal{V}_\nu$  and  $\mathcal{V}_\omega$  are as in the paragraph before Theorem 79.

Let  $\Delta^* \subset \Delta^{*r}$  be the punctured disk defined by the equation  $s_1 = \cdots = s_r$ , and fix a point  $s_0 \in \Delta^*$ . Let  $V$  be the fiber of  $\mathcal{V}_\nu$  over  $s_0$ . Then, applying Corollary 86 to the restriction of  $\mathcal{V}_\nu$  to  $\Delta^* \subset \Delta^{*r}$  produces a trigrading  $V_k^{p,q}$  of  $V$  which is split over  $\mathbb{R}$ . Likewise, let  $U$  be the fiber of  $\mathcal{V}_\omega^\vee$  over  $s_0$ . Then, using duality, restriction of  $\mathcal{V}_\omega^\vee$  to  $\Delta^* \subset \Delta^{*r}$  produces a trigrading  $U_k^{p,q}$  of  $U$  which is also split over  $\mathbb{R}$ .

Since  $U$  and  $V$  abut at  $\ell = -1$ , we can form the trigraded vector space

$$B = U \star V$$

with associated trigrading which is split over  $\mathbb{R}$ . The associated bigrading determines an  $\mathbb{R}$ -split mixed Hodge structure  $(\hat{F}_\infty, M) = (\hat{F}_\infty(B), M(B))$  as in (80).

Let  $(F_\nu, M^\nu)$  be the limit mixed Hodge structure of  $\mathcal{V}_\nu$  and  $(F_\omega, M^\omega)$  be the limit mixed Hodge structure  $\mathcal{V}_\omega^\vee$ . Let  $\zeta_\nu \in \mathfrak{gl}(V)$  and  $\zeta_\omega \in \mathfrak{gl}(U)$  be the corresponding endomorphisms (56) of  $V$  and  $U$ . Recall that  $\zeta_\nu$  and  $\zeta_\omega$  preserve the weight filtrations  $W(V)$  and  $W(U)$  respectively. Moreover, they induce the same action on  $Gr_{-1}^W$  since  $\mathcal{V}_\nu$  and  $\mathcal{V}_\omega^\vee$  induce the same limit mixed Hodge structure on  $Gr_{-1}^W$ . Define

$$F_\infty = e^{\zeta} \hat{F}_\infty \tag{88}$$

where  $\zeta = \zeta_U \star \zeta_V$ . Since  $\zeta_\nu$  and  $\zeta_\omega$  act trivially on  $Gr^M(V)$  and  $Gr^M(U)$  it follows that  $\zeta$  acts trivially on  $Gr^M$  by (84).

*Remark 89.* The mixed Hodge structure  $(\hat{F}_\infty, M)$  projects to  $(\hat{F}_\nu, M^\nu)$  and  $(\hat{F}_\omega, M^\omega)$  on  $V$  and  $U$  respectively. The monodromy logarithms  $N_j^V$  and  $N_j^U$  of  $\mathcal{V}_\nu$  and  $\mathcal{V}_\omega^\vee$  are  $(-1, -1)$ -morphisms of  $(\hat{F}_\nu, M^\nu)$  and  $(\hat{F}_\omega, M^\omega)$ . For future use, we also note that  $W_{-2} \text{End}(B)$  consists of  $(-1, -1)$ -morphisms of  $(\hat{F}_\infty, M)$  and  $(F_\infty, M)$ . This follows from the fact that  $B_0 = B_0^{0,0}$  while  $B_{-2} = B_{-2}^{-1,-1}$ .

**Local System.** The monodromy logarithms  $N_j^V$  and  $N_j^U$  induce the same action on  $Gr_{-1}^W$  and hence glue together to define an endomorphism  $\tilde{N}_j$  of  $B_\mathbb{R}$ . By the previous remark,  $\tilde{N}_j$  is a  $(-1, -1)$ -morphism of  $(\hat{F}_\infty, M)$ . To show that the endomorphisms  $\{\tilde{N}_j\}$  define a local system over  $\Delta^{*r}$  with fiber  $B_\mathbb{R}$ , it is sufficient to show that  $[\tilde{N}_j, \tilde{N}_k] = 0$  for all  $j$  and  $k$ .

Let  $\{V_i^{p,q}\}$  be the trigrading of  $V$  of Corollary (86), and let  $v_0 \in V_0$  be a lifting of  $1 \in Gr_0^W$ . Set  $b_0 = \iota_V(v_0)$ . Then, since  $v_0$  is type  $(0, 0)$  with respect to the bigrading of  $V$ , the same is true of  $b_0$  with respect to the bigrading of  $B$ . Moreover, tracing through the above definitions, one sees that

$$\pi_V[\tilde{N}_j, \tilde{N}_k]b_0 = [N_j^V, N_k^V]v_0 = 0$$

Therefore,  $[\tilde{N}_j, \tilde{N}_k]b_0$  is an element of  $W_{-2}(B)$  which is of type  $(-2, -2)$  with respect to the bigrading of  $B$ . But  $W_{-2}(B)$  is pure of type  $(-1, -1)$ , and hence  $[\tilde{N}_j, \tilde{N}_k]b_0 = 0$ . Likewise,  $\pi_U[\tilde{N}_j, \tilde{N}_k]\iota_U(u) = [N_j^U, N_k^U]u = 0$ . Consequently,  $[\tilde{N}_j, \tilde{N}_k] = 0$ .

**A-Structure.** Pick an element  $v_A \in V_A$  which projects to  $1 \in Gr_0^W(V)$  and let  $b_A = \iota_V(v_A)$ . Define

$$B_A = Ab_A \oplus \iota_U(U_A)$$

Then,  $B_A \otimes \mathbb{R} = B_{\mathbb{R}}$  since  $v_A = v_0 + v_{-1}$  with  $v_0, v_{-1}$  real and  $v_0$  as above in  $V_0^{0,0}$ .

Let  $u_{-2}$  be the generator of  $W_{-2}(U_A) \cong A(1)$  corresponding to  $1^\vee$  (with  $1^\vee$  a generator of  $A(1)$  as in Definition 36). Set  $b_{-2} = \iota_U(u_{-2})$ . Note that  $b_{-2} \in B_{-2}^{-1,-1}$ . Let  $\eta$  be the endomorphism of  $B_{\mathbb{R}}$  which annihilates  $W_{-1}(B)$  and maps  $b_A$  to  $b_{-2}$  (or, equivalently,  $b_0$  to  $b_{-2}$ ). We are going to set

$$N_j = \tilde{N}_j + c_j \eta \quad (90)$$

for some scalar  $c_j$  to be determined below.

Since  $W_{-2}(B)$  has rank 1 and  $\tilde{N}_j$  is nilpotent and preserves  $W(B)$ , it follows that  $\tilde{N}_j$  acts trivially on  $W_{-2}(B)$ . Consequently,  $[\tilde{N}_j, \eta] = 0$  and hence  $[N_j, N_k] = 0$ . Likewise,  $[N_j, \eta] = 0$  and hence  $T_j = e^{N_j} = e^{\tilde{N}_j + c_j \eta} = (\tilde{T}_j)(1 + c_j \eta)$ . Therefore,

$$T_j(b_A) = \tilde{T}_j(b_A) + c_j u_{-2} = b_A + (\tilde{T}_j - 1)b_A + c_j u_{-2}$$

In particular, because  $(T_j^V - 1)v$  maps to an integral class in  $Gr_{-1}^W$ , it follows that for suitable choice of real scalar  $c_j$ ,  $T_j(b) \in B_A$ . Likewise,  $T_j$  acts as  $T_j^U$  on  $U$ , and hence  $T_j$  preserves  $B_A$ .

**Hodge filtration.** By rescaling  $s$  if necessary, we can assume that  $\mathcal{V}_v$  and  $\mathcal{V}_\omega^\vee$  have local normal forms  $\Gamma^V$  and  $\Gamma^U$  on  $\Delta^r$  as in (46). Then,  $\Gamma^V$  preserves  $W(V)$ ,  $\Gamma^U$  preserves  $W(U)$  and the induced actions on  $Gr_{-1}^W$  agree since the induced VHS on  $Gr_{-1}^W$  coincide. Therefore, we can glue  $\Gamma^V$  to  $\Gamma^U$  to define a function  $\Gamma$  on  $\Delta^r$  which preserves  $W(U \star V)$  and acts by infinitesimal isometries on  $Gr^W B$ . Define

$$F(z) = e^{\sum_j z_j N_j} e^{\Gamma(s)} F_\infty \quad (91)$$

where  $F_\infty$  is the filtration (88), and  $N_j$  is the endomorphism (90). Then,  $F(z)$  induces the corresponding Hodge filtrations of  $\mathcal{V}_v$  on  $Gr_j^W(V)$  for  $j = 0, j = -1$ , and  $\mathcal{V}_\omega^\vee$  for  $j = -1, -2$ . Clearly,  $F(z)$  is holomorphic and descends to a period map over  $\Delta^{*r}$  with monodromy  $\{T_j\}$ .

**Horizontality.** We need to show that  $\frac{\partial}{\partial z_j} F^p(z) \subseteq F^{p-1}(z)$ . By manipulation of (91), this is equivalent to

$$\left( e^{-\text{ad} \Gamma} N_j + 2\pi i s_j e^{-\Gamma} \frac{\partial}{\partial s_j} e^{\Gamma} \right) (F_\infty^p) \subseteq F_\infty^{p-1}$$

By construction, this holds modulo  $W_{-2} \text{End}(B)$ , which is sufficient to prove horizontality for  $p > 0$ . For the case  $p = 0$ , observe that  $W_{-2} \text{End}(B)$  consists of  $(-1, -1)$ -morphisms of  $(F_\infty, M)$ . For  $p < 0$ , horizontality follows from horizontality modulo  $W_{-2}B$  and the case  $p = 0$ .

**Admissibility.** To show that (91) defines an admissible normal function, it remains to show that (i) the Hodge bundles extend holomorphically over  $\Delta^r$  and induces the limit filtration of Schmid on each  $Gr^W$ , and (ii) the required relative weight filtrations exist.

The first condition follows from that fact that  $\exp(-\sum_j z_j N_j) F(z) = e^{\Gamma(s)} F_\infty$  which obviously extends to a holomorphic filtration on  $\Delta^r$ , and  $F_\infty$  induces Schmid's filtration on  $Gr^W$  by construction.

To prove the existence of the required relative weight filtrations, we recall the following result from [39, Theorem 2.20]: Let  $(i)M$  denote the relative weight filtration of  $N_{W_i}$  and  $W$ . Suppose that  $(k-1)M$  exists. Then,  $(k)M$  exists if and only if for all  $j > 0$ ,

$$(N^j W_j) \cap W_{k-1} \subset N^j W_{k-1} + (k-1)M_{k-j-1}$$

Set  $N = N_j$  and  $W = W(B)$ . Then, the relative weight filtration of  $N$  restricted to  $W_{-1}$  exists and equals the image of  $M(N_j^U, W(U))$  under  $\iota_U$ . Therefore, we need only check the above for  $k = 0$ . Since  $N = \tilde{N}_j + c_j \eta$ ,

$$(N^j W_j) \cap W_{-1} = N^j W_{-1} + \text{Im}(c_j \eta) \subset N^j W_{-1} + (-1)M_{-2}$$

because  $W_{-2} \subset (-1)M_{-2}$  since  $W_{-2}$  is pure of type  $(-1, -1)$  for the limit mixed Hodge structure.

This completes the proof of Theorem 79.

**Quasi-Unipotent Monodromy.** In this section, we extend Theorem 79 to the case of admissible normal functions with quasi-unipotent monodromy, and give an analog of the local normal form (46) for admissible variations with quasi-unipotent monodromy.

Let  $T$  be an automorphism of a finite dimensional complex vector space. Then, by the Jordan decomposition theorem,  $T = T_s + T_n$  where  $T_s$  is semisimple,  $T_n$  is nilpotent and  $[T_s, T_n] = 0$ . Moreover, there exists polynomials  $p$  and  $q$  without constant term such that  $T_s = p(T)$  and  $T_n = q(T)$ . In particular, since  $T$  is an automorphism,  $T_s$  is invertible. Let  $T_u = 1 + T_s^{-1}T_n$ . Then,  $T_u$  is unipotent and  $T = T_s T_u = T_u T_s$  is called the multiplicative Jordan decomposition of  $T$ .

*Remark 92.* If  $T$  is a automorphism of a finite dimensional rational vector space, and  $T_s$  has finite order  $m$ , then both  $T_s$  and  $T_u$  are rational. To see this, note that  $T_u = e^\alpha$  and hence  $T^m = T_u^m = e^{m\alpha}$  is rational. Therefore,  $\alpha$  is rational, and hence so is  $T_u$  and  $T_s = T T_u^{-1}$ .

**Proposition 93.** *Let  $T$  and  $T'$  be commuting automorphisms of a finite dimensional complex vector space, and let  $T = T_s + T_n$  and  $T' = T'_s + T'_n$  be the Jordan decomposition of  $T$  and  $T'$ . Then,  $\{T_s, T_n, T'_s, T'_n\}$  is a set of mutually commuting endomorphisms.*

*Proof.* Each of the elements listed is a polynomial in  $T$  or  $T'$ . □

**Corollary 94.** *Let  $T$  and  $T'$  be commuting automorphisms of a finite dimensional complex vector space, and let  $T = T_s T_u$  and  $T' = T'_s T'_u$  be the multiplicative Jordan decompositions of  $T$  and  $T'$ . Then,  $\{T_s, T_u, T'_s, T'_u\}$  is a set of mutually commuting automorphisms.*

Let  $(\Delta^r, s)$  be a polydisk with holomorphic coordinates  $(s_1, \dots, s_r)$  and  $\mathcal{V} \rightarrow \Delta^{*r}$  be a variation of mixed Hodge structure defined on the complement of the divisor  $s_1 \cdots s_r = 0$ . Assume that  $\mathcal{V}$  has quasi-unipotent monodromy  $\gamma_i$  about  $s_i = 0$ , and let  $\gamma_i = \gamma_{i,s} \gamma_{i,u}$  be the multiplicative Jordan decomposition of  $\gamma_i$ . Then, by the previous Corollary,  $\{\gamma_{j,s}, \gamma_{j,u}, \gamma_{k,s}, \gamma_{k,u}\}$  is a set of mutually commuting automorphisms of the reference fiber of  $\mathcal{V}$  for any pair of indices  $j$  and  $k$ . Let  $m_i > 0$  denote the order of  $\gamma_{i,s}$  and  $N_i = \log \gamma_{i,u}$ .

Assume that  $\mathcal{V} \rightarrow (\Delta^{*r}, s)$  is admissible. Let  $(\Delta^{*r}, t)$  be a polydisk with coordinates  $(t_1, \dots, t_r)$  and  $f : (\Delta^{*r}, t) \rightarrow (\Delta^{*r}, s)$  denote the covering map  $t_j = s_j^{m_j}$ . Then,  $f^*(\mathcal{V})$  is an admissible variation of mixed Hodge structure over  $(\Delta^{*r}, t)$  with unipotent monodromy  $e^{m_j N_j}$  about  $t_j = 0$ . Let

$$\tilde{F} : (\mathfrak{h}^r, w) \rightarrow \mathcal{M}$$

denote the lifting of the period map of  $f^*(\mathcal{V})$  to the product of upper half-planes  $\mathfrak{h}^r \subset \mathbb{C}^r$  with Cartesian coordinates  $(w_1, \dots, w_r)$  relative to the covering map  $t_j = e^{2\pi i w_j}$ .

For  $w \in \mathbb{C}^r$  let  $N(w) = \sum_k w_k m_k N_k$ , and

$$\tilde{\psi}(w) = e^{-N(w)} \tilde{F}(w). \quad (95)$$

Let  $\{e_1, \dots, e_r\}$  denote the standard Cartesian basis of  $\mathbb{C}^r$ . Then,  $\tilde{\psi}(w + e_j) = \tilde{\psi}(w)$ , and hence  $\tilde{\psi}$  descends to a map

$$\psi : \Delta^{*r} \rightarrow \check{\mathcal{M}}.$$

By admissibility,  $\psi$  extends to a holomorphic map  $\Delta^r \rightarrow \check{\mathcal{M}}$ , such that

$$(m_1 N_1, \dots, m_r N_r; F_\infty, W) \quad (96)$$

is an admissible nilpotent orbit, where  $F_\infty = \psi(0)$ .

Let  $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p,q < 0} \mathfrak{g}^{p,q}$  relative to the limit mixed Hodge structure  $(F_\infty, M)$  of (96). Then,

$$\mathfrak{q} = \bigoplus_{a < 0, b} \mathfrak{g}^{a,b}$$

is a vector space complement to the stabilizer of  $F_\infty$  in  $\mathfrak{g}_{\mathbb{C}}$ . Therefore, there exists a unique holomorphic function

$$\Gamma : \Delta^r \rightarrow \mathfrak{q}$$

such that  $e^{\Gamma(t)} F_\infty = \psi(t)$ , after shrinking  $\Delta^r$  as necessary. This gives the local normal form

$$\tilde{F}(w) = e^{N(w)} e^{\Gamma(t)} F_\infty. \quad (97)$$

The next result asserts that each  $\gamma_{j,s}$  is a morphism of the limit mixed Hodge structure. In the geometric case this is due to J. Steenbrink [38].

**Proposition 98.** *Each  $\gamma_{j,s}$  is morphism of  $(F_\infty, M)$ . Moreover,*

$$\Gamma \circ \rho_j(t) = \text{Ad}(\gamma_{j,s}) \Gamma(t) \quad (99)$$

where  $\rho_j(t_1, \dots, t_r) = (t_1, \dots, e^{2\pi i/m_j} t_j, \dots, t_r)$ .

*Proof.* To prove that  $\gamma_{j,s}$  is a morphism of the limit mixed Hodge structure we begin with the observation that

$$\psi \circ \rho_j(t) = \gamma_{j,s} \psi(t) \quad (100)$$

To see this we note that we have a commutative diagram

$$\begin{array}{ccccc} (\mathfrak{h}^r, w) & \xrightarrow{z_k = m_k w_k} & (\mathfrak{h}^r, z) & \xrightarrow{F(z)} & \mathcal{M} \\ t_k = e^{2\pi i w_k} \downarrow & & \downarrow s_k = e^{2\pi i z_k} & & \downarrow \\ (\Delta^{*r}, t) & \xrightarrow{s_k = t_j^{m_k}} & (\Delta^{*r}, s) & \xrightarrow{\varphi} & \Gamma \backslash \mathcal{M} \end{array}$$

where  $F : (\mathfrak{h}^r, z) \rightarrow \mathcal{M}$  is the local lifting of the period map of  $\mathcal{V} \rightarrow (\Delta^{*r}, s)$  relative to the covering map  $s_k = e^{2\pi i z_k}$  where  $(z_1, \dots, z_r)$  are Cartesian coordinates on the product of upper half-planes  $\mathfrak{h}^r \subset \mathbb{C}^r$ . In particular, since  $F(z + e_j) = \gamma_j F(z)$  it follows that  $\tilde{F}(w + (1/m_j)e_j) = \gamma_j \tilde{F}(w)$ . Therefore,

$$\psi \circ \rho_j(t) = e^{-N(w + (1/m_j)e_j)} \tilde{F}(w + (1/m_j)e_j) = e^{-N(w + (1/m_j)e_j)} \gamma_j \tilde{F}(w) = \gamma_{j,s} \psi(t).$$

Taking the limit as  $t \rightarrow 0$  yields  $F_\infty = \gamma_{j,s} F_\infty$ .

By the remark at the beginning of this section, in the case where  $A = \mathbb{Z}$  or  $A = \mathbb{Q}$ ,  $\gamma_{j,s}$  is rational since  $\gamma_{j,s}$  is finite order and  $\gamma_j$  is rational. To prove that  $\gamma_{j,s}$  is a morphism of  $(F, M)$  it remains to check that  $\gamma_{j,s}$  preserves  $M$ . This follows from the defining properties of the relative weight filtration and the fact that:

- Since  $\gamma_j$  preserves  $W$  so does  $\gamma_{j,s} = p(\gamma_j)$ ;
- $\gamma_{j,s}$  commutes with  $N_1, \dots, N_r$  and  $M = M(N_1 + \dots + N_r, W)$ .



To verify (99), note that by the derivation of (97) and equation (100), we have

$$\psi \circ \rho_j(t) = e^{\Gamma \circ \rho_j(t)} F_\infty = \gamma_{j,s} \psi(t) = \gamma_{j,s} e^{\Gamma(t)} F_\infty.$$

Since  $\gamma_{j,s}$  preserves  $F_\infty$  by the previous paragraphs, it follows that

$$e^{\Gamma \circ \rho_j(t)} F_\infty = \gamma_{j,s} e^{\Gamma(t)} \gamma_{j,s}^{-1} F_\infty.$$

In particular, since  $\gamma_{j,s}$  is a morphism of  $(F_\infty, M)$  the adjoint action of  $\gamma_{j,s}$  preserves  $\mathfrak{q}$ . Because  $\mathfrak{q}$  is a vector space complement to the stabilizer of  $F_\infty$  in  $\mathfrak{g}_\mathbb{C}$ , it then follows that  $\Gamma \circ \rho_j(t) = \text{Ad}(\gamma_{j,s})\Gamma(t)$ .  $\square$

To continue, we note that by commutative diagram of Proposition (98), we have  $\tilde{F}(w_1, \dots, w_r) = F(m_1 w_1, \dots, m_r w_r)$ . Setting  $w_j = z_j/m_j$  it then follows from (97) that

$$F(z_1, \dots, z_r) = \tilde{F}(z_1/m_1, \dots, z_r/m_r) = e^{\sum_k z_k N_k} e^{\Gamma(v_1, \dots, v_r)} F_\infty \quad (101)$$

where  $v_j = e^{2\pi i z_j/m_j}$ .

**Theorem 102.** *Let*

- (1)  $\theta(z_1, \dots, z_r) = (e^{\sum_k z_k N_k} F_\infty, W)$  be an admissible nilpotent orbit with limit mixed Hodge structure  $(F_\infty, M)$  and values in the a classifying space  $\mathcal{M}$  of graded-polarized mixed Hodge structure for  $\text{Im}(z_1), \dots, \text{Im}(z_r) \gg 0$ ;
- (2)  $\{\gamma_{1,s}, \dots, \gamma_{r,s}\}$  be a set of semisimple automorphisms of  $(F_\infty, M)$  which act by finite order isometries on  $\text{Gr}^W$ . Let  $m_j$  be the order of  $\gamma_{j,s}$  and  $\gamma_{j,u} = e^{N_j}$ . Assume that  $\{\gamma_{1,s}, \gamma_{1,u}, \dots, \gamma_{r,s}, \gamma_{r,u}\}$  is a set of mutually commuting automorphisms;
- (3)  $\Gamma : \Delta^r \rightarrow \mathfrak{q}$  be a holomorphic function which vanishes at the origin and satisfies (99).

Suppose that

$$\tilde{F}(w_1, \dots, w_r) = e^{\sum_k w_k m_k N_k} e^{\Gamma(t)} F_\infty \quad (103)$$

satisfies Griffiths infinitesimal period relation on the upper half-plane  $(\mathfrak{h}^r, w)$  where  $t_j = e^{2\pi i w_j}$ . Define  $F(z)$  from  $\tilde{F}(w)$  via equation (101). Then,

- (i)  $F(z_1, \dots, z_j + 1, \dots, z_r) = \gamma_i F(z_1, \dots, z_r)$  where  $\gamma_i = \gamma_{i,s} \gamma_{i,u}$ ;
- (ii) There exists a positive real number  $A$  such that

$$F(z_1, \dots, z_r) = e^{\sum_j z_j N_j} e^{\Gamma(v_1, \dots, v_r)} F_\infty$$

descends to an admissible period map on the set of points in  $(\Delta^{*r}, s)$  such that  $|s_j| = |e^{2\pi i z_j}| < A$ .

*Proof.* Since  $v_j = e^{2\pi i z_j/m_j}$  it follows that changing  $z_j$  to  $z_j + 1$  changes  $v_j$  to  $e^{2\pi i/m_j} v_j$ . Therefore, by equation (99),

$$\begin{aligned} F(z_1, \dots, z_j + 1, \dots, z_r) &= e^{N_j} e^{\sum_k z_k N_k} \gamma_{j,s} e^{\Gamma(v_1, \dots, v_r)} \gamma_{j,s}^{-1} F_\infty \\ &= \gamma_{j,u} \gamma_{j,s} e^{\sum_k z_k N_k} e^{\Gamma(v_1, \dots, v_r)} F_\infty \end{aligned}$$

using hypothesis (2) and (3). This proves part (i).

To prove part (ii), we need to show that

- (a)  $F(z)$  satisfies Griffiths infinitesimal period relation;
- (b)  $F(z)$  takes values in  $\mathcal{M}$  for  $\text{Im}(z_1), \dots, \text{Im} z_r \gg 0$ ;
- (c) Verify the existence of the requisite relative weight filtrations.

Condition (c) follows from the fact that  $\theta$  is an admissible nilpotent orbit. On the other hand, since  $(\Delta^{*r}, t) \rightarrow (\Delta^{*r}, s)$  is a covering map, it is sufficient to verify conditions (a) and (b) for the map (103). Condition (a) holds for  $\tilde{F}(w)$  by assumption. To verify condition (b), we need only show that  $\tilde{F}(w)$  induces pure, polarized Hodge structures on  $\text{Gr}^W$  for  $\text{Im}(w_1), \dots, \text{Im} w_r \gg 0$ . This

follows immediately from the fact that  $\theta$  is an admissible nilpotent orbit,  $\tilde{F}(w)$  is horizontal and Theorem (2.8) [8].  $\square$

*Remark 104.* In order to produce a variation with an integral (or rational structure) via Theorem (102), each  $\gamma_{j,s}$  and  $\gamma_{j,u}$  must be integral (or rational).

**Theorem 105.** *Theorem (79) remains valid in the setting where  $\mathcal{H}$  has quasi-unipotent monodromy.*

*Proof.* By Theorem (102), it is sufficient to work on the pullback to the unipotent case and ensure that the resulting function  $\Gamma$  satisfies (99). Each of the functions  $\Gamma^V$  and  $\Gamma^U$  attached to the normal functions satisfy an appropriate version of (99) relative to the semisimple automorphisms  $\gamma_{j,s}^V$  and  $\gamma_{j,s}^U$  respectively. It follows from Proposition (87) that  $\gamma_{j,s}^V$  and  $\gamma_{j,s}^U$  preserve the trigradings of  $V$  and  $U$ . Since they agree on  $Gr_{-1}^W$  it follows that they glue together to an automorphism  $\gamma_{j,s}$  of  $B$  which preserves the trigrading of  $B$ . Therefore, the function  $\Gamma$  produced by gluing  $\Gamma^U$  and  $\Gamma^V$  also satisfies (99).

In the case where  $A = \mathbb{Z}$  or  $\mathbb{Q}$ , we add the underlying  $A$  structure as follows: The element  $\eta$  appearing (90) commutes with each  $\gamma_{j,s}$  since  $\gamma_{j,s} \in gl(B)_0^{0,0}$  with respect to the induced trigrading on  $gl(B)$ , and  $Gr(\gamma_{j,s})$  acts trivially on the 1-dimensional factors  $Gr_0^W$  and  $Gr_{-2}^W$  by hypothesis (i.e. these factors are assumed to have trivial global monodromy). The construction of the  $A$ -structure therefore follows exactly as in the unipotent case.  $\square$

## 5. SINGULARITIES OF NORMAL FUNCTIONS

In this section, we review the notion of the singularities of normal functions, in the setting where  $\mathcal{H}$  is an admissible variation of mixed Hodge structure such that  $W_{-1}\mathcal{H} = \mathcal{H}$ .

**5.1. Admissible Normal Functions.** Suppose  $\bar{S}$  is a complex manifold and  $S$  is a Zariski open subset. Suppose  $\mathcal{H}$  is an object in  $VMHS(S)_{\bar{S}}^{\text{ad}}$  with  $W_{-1}\mathcal{H} = \mathcal{H}$  and underlying  $\mathbb{Z}$ -local structure  $\mathcal{H}_{\mathbb{Z}}$ . Following M. Saito, we define

$$\text{ANF}(S, \mathcal{H})_{\bar{S}} := \text{Ext}_{VMHS(S)_{\bar{S}}^{\text{ad}}}^1(\mathbb{Z}, \mathcal{H})$$

to be the group of admissible normal functions on  $S$  relative to  $\bar{S}$  with underlying variation of mixed Hodge structure  $\mathcal{H}$ .

**5.2. Singularities of admissible normal functions.** Let  $S, \bar{S}$  and  $\mathcal{H}$  be as in §5.1, and let  $s \in \bar{S} \setminus S$ . Write  $j : S \rightarrow \bar{S}$  and  $i : \{s\} \rightarrow \bar{S}$  for the inclusions. Suppose  $\nu \in \text{ANF}(S, \mathcal{H})_{\bar{S}}$  is an admissible normal function. Let

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{V} \rightarrow \mathbb{Z} \rightarrow 0 \tag{106}$$

by the extension class corresponding to  $\nu$ . By applying the forgetful functor from  $VMHS(S)_{\bar{S}}^{\text{ad}}$  to the category  $\text{Loc}(S)$  of local systems on  $S$ , we wind up with an extension in the category of local systems on  $S$ . Thus we can associate to  $\nu$  a class  $\text{cl}(\nu) \in H^1(S, \mathcal{H})$ . Tensoring with  $\mathbb{Q}$ , we get a class  $\text{cl}_{\mathbb{Q}} \in H^1(S, \mathcal{H}_{\mathbb{Q}}) = H^1(\bar{S}, Rj_*\mathcal{H}_{\mathbb{Q}})$ . Finally, by pull-back via  $i$ , we get a class

$$\text{sing}_s(\nu) \in H_s^1(\mathcal{H}_{\mathbb{Q}}) := H^1(i^*Rj_*\mathcal{H}_{\mathbb{Q}}). \tag{107}$$

This class is called the *singularity* of  $\nu$ . We write  $\text{cl}_s(\nu)$  for the image of  $\text{cl}(\nu)$  in  $H^1(i^*Rj_*\mathcal{H})$ .

**5.3. Singularities.** Suppose  $\mathbb{F} = \mathbb{Q}$  or  $\mathbb{R}$ . Let  $\mathcal{L}$  denote a  $\mathbb{F}$ -local system on  $S$  and  $d = \dim_{\mathbb{C}} S$ . The intersection complex of  $\mathcal{L}$  is

$$\mathrm{IC}(\mathcal{L}) := j_{!*}(\mathcal{L}[d])$$

The intersection and local intersection cohomology spaces are defined by

$$\mathrm{IH}^k(\bar{S}, \mathcal{L}) := H^{k-d}(\bar{S}, \mathrm{IC}(\mathcal{L}))$$

$$\mathrm{IH}_s^k(\mathcal{L}) := H^{k-d}(i^* \mathrm{IC}(\mathcal{L})).$$

There is a morphism  $\mathrm{IC}(\mathcal{L}) \rightarrow Rj_*\mathcal{L}$  which induces maps  $\mathrm{IH}^k(\bar{S}, \mathcal{L}) \rightarrow H^k(S, \mathcal{L})$  and  $\mathrm{IH}_s^k(\mathcal{L}) \rightarrow H_s^k(\mathcal{L}) := H^k(i^* Rj_*\mathcal{H}_Q)$ . For  $k = 0$ , these maps are isomorphisms, and for  $k = 1$  they are injections.

Note that, by M. Saito's theory of mixed Hodge modules, if  $\mathcal{H}$  is a  $\mathbb{Q}$  variation of mixed Hodge structure, then  $H_s^p(\mathcal{H})$  and  $\mathrm{IH}_s^p(\mathcal{H})$  both carry canonical mixed Hodge structures.

**Theorem 108.** [3] *Suppose  $\mathcal{H}$  is a weight  $-1$  variation of Hodge structure on  $S$ . The map*

$$\mathrm{cl}_Q : \mathrm{ANF}(S, \mathcal{H})_{\bar{S}} \rightarrow H^1(S, \mathcal{H}_Q)$$

*factors through  $\mathrm{IH}^1(S, \mathcal{H}_Q)$ . Similarly, the map*

$$\mathrm{sing}_s : \mathrm{ANF}(S, \mathcal{H})_{\bar{S}} \rightarrow H_s^1(\mathcal{H}_Q)$$

*factors through  $\mathrm{IH}_s^1(\mathcal{H}_Q)$ .*

**5.4. Duality in the Weight  $-1$  case.** If  $\mathcal{H}$  is variation of pure Hodge structure of weight  $-1$  on  $S$ , as in the introduction, we write  $\mathcal{H}^\vee := \mathcal{H}^*(1)$ . This is another variation of pure Hodge structure of weight  $-1$  on  $S$ . There is a canonical morphism  $\mathcal{H} \rightarrow (\mathcal{H}^\vee)^\vee$  which is an isomorphism if  $\mathcal{H}$  is torsion-free.

**Proposition 109.** *Suppose  $\mathcal{H}$  is torsion free. Then duality gives a canonical isomorphism*

$$\mathrm{ANF}(S, \mathcal{H})_{\bar{S}} \cong \mathrm{Ext}_{\mathrm{VMHS}(S)_{\bar{S}}}^1(\mathcal{H}^\vee, \mathbb{Z}(1))$$

*Proof.* We have

$$\mathrm{ANF}(S, \mathcal{H})_{\bar{S}} = \mathrm{Ext}_{\mathrm{VMHS}(S)_{\bar{S}}}^1(\mathbb{Z}, \mathcal{H}) \xrightarrow{D} \mathrm{Ext}_{\mathrm{VMHS}(S)_{\bar{S}}}^1(\mathcal{H}^*, \mathbb{Z}) = \mathrm{Ext}_{\mathrm{VMHS}(S)_{\bar{S}}}^1(\mathcal{H}^\vee, \mathbb{Z}(1)).$$

The map  $D$ , which is given by duality, is an isomorphism because  $\mathcal{H}$  is torsion-free.  $\square$

### 5.5. Admissible classes.

**Definition 110.** Let  $j : S \rightarrow \bar{S}$  and  $i : \{s\} \rightarrow \bar{S}$  be as in 5.1. We call a holomorphic map  $\bar{\varphi} : \Delta \rightarrow \bar{S}$  a *test curve at  $s$*  if  $\bar{\varphi}(0) = s$  and  $\bar{\varphi}(\Delta^*) \subset S$ . Write  $\varphi : \Delta^* \rightarrow S$  for the restriction of  $\bar{\varphi}$  to  $\Delta^*$ . Suppose  $\mathcal{L}$  is a  $\mathbb{F}$ -local system on  $S$  for  $\mathbb{F} = \mathbb{Q}$  or  $\mathbb{R}$  (as in §5.2). We call a class  $\alpha \in H_s^1(\mathcal{L})$  *admissible* if the pull-back  $\varphi^*\alpha$  in  $H_0^1(\varphi^*\mathcal{L})$  vanishes for every test curve through  $s$ . Similarly, we call a class  $\alpha \in H^1(S, \mathcal{L})$  *admissible* if  $\varphi^*\alpha$  vanishes in  $H^1(\Delta^*, \varphi^*\mathcal{L})$  for all test curves through all points of  $\bar{S}$ . We write  $H_s^1(\mathcal{L})^{\mathrm{ad}}$  (resp.  $H^1(S, \mathcal{L})^{\mathrm{ad}}$ ) for the set of admissible classes in  $H_s^1(\mathcal{L})$  (resp.  $H^1(S, \mathcal{L})$ ). Similarly, we write  $\mathrm{IH}_s^1(\mathcal{L})^{\mathrm{ad}} := \mathrm{IH}_s^1(\mathcal{L}) \cap H_s^1(\mathcal{L})^{\mathrm{ad}}$  and  $\mathrm{IH}^1(\bar{S}, \mathcal{L})^{\mathrm{ad}} := \mathrm{IH}^1(\bar{S}, \mathcal{L}) \cap H^1(S, \mathcal{L})^{\mathrm{ad}}$ .

**Theorem 111.** *Let  $\mathcal{L}$  be an variation of pure Hodge structure on  $S$ . Then for  $s \in \bar{S}$ , we have  $\mathrm{IH}_s^1(\mathcal{L}) = \mathrm{IH}_s^1(\mathcal{L})^{\mathrm{ad}}$ . In other words, every class is admissible.*

We will prove the theorem after the following lemma.

**Lemma 112.** *Suppose  $\mathcal{L}$  is a variation of pure Hodge structure of weight  $w$  on  $\Delta^*$ . Then  $\mathrm{Gr}_k^W H_0^1(\mathcal{L}) = 0$  for  $k < w + 2$ . In other words,  $H_0^1\mathcal{L}$  has weights in the interval  $[w + 2, \infty)$ .*

*Proof of Lemma 112.* We have an exact sequence

$$0 \rightarrow H_0^0(\mathcal{L}) \rightarrow \psi_{z,1}\mathcal{L} \xrightarrow{N} \psi_{z,1}\mathcal{L}(-1) \rightarrow H_0^1(\mathcal{L}) \rightarrow 0 \quad (113)$$

where  $\psi_{z,1}$  denotes the unipotent nearby cycles functor and  $N$  denotes the logarithm of unipotent part of the monodromy (of  $\mathcal{L}$  around 0). The weight filtration on  $\psi_{z,1}\mathcal{L}$  is induced from the monodromy filtration of  $N$  so that

$$N^k : \mathrm{Gr}_{w-k}^W(\psi_{z,1}\mathcal{L}(k)) \rightarrow \mathrm{Gr}_{w-k}^W \psi_{z,1}\mathcal{L}$$

is an isomorphism for  $k \geq 0$ . So  $\mathrm{coker}(N : \psi_{z,1}\mathcal{L}(1) \rightarrow \psi_{z,1}\mathcal{L})$  has weights in the interval  $[w, \infty)$ . The result follows immediately.  $\square$

*Proof of Theorem 111.* Suppose  $\bar{\varphi} : \Delta \rightarrow \bar{S}$  is a test curve through  $s$ . Write  $f$  for the composition

$$\mathrm{IH}_s^1(\mathcal{L}) \rightarrow H_s^1(\mathcal{L}) \rightarrow H_0^1(\varphi^*\mathcal{L}).$$

Then  $f$  is a morphism of mixed Hodge structures. By the purity theorem of Cattani-Kaplan-Schmid and Kashiwara-Kawai, the source of  $f$  has weights in the interval  $(-\infty, w+1]$ . But by Lemma 112, the target has weights in the interval  $[w+2, \infty)$ . It follows that  $f = 0$ .  $\square$

## 6. THE ASYMPTOTIC HEIGHT PAIRING IN THE NORMAL CROSSING CASE

The goal of this section is to define the asymptotic height pairing on local intersection cohomology in the normal crossing case using a complex  $B^*$  which computes the local intersection cohomology in this case. (See Theorem 133.)

**6.1. Normal crossing setup.** In this section, we write  $\mathbb{F}$  for a field which is either  $\mathbb{Q}$  or  $\mathbb{R}$  and  $r$  for a fixed non-negative integer. Let  $\Delta^r$  denote a polydisk with local coordinates  $(s_1, \dots, s_r)$  and  $\Delta^{*r}$  be the complement of the divisor  $s_1 \cdots s_r = 0$ . Let  $\mathcal{L}$  be a local system of  $\mathbb{F}$ -vector spaces on  $\Delta^{*r}$  with unipotent monodromy, and we write  $L$  for a fixed fiber of  $\mathcal{L}$ . We let  $T_1, \dots, T_r \in \mathrm{Aut}_{\mathbb{F}} L$  denote the monodromy operators. Then, for each  $i$ , the monodromy logarithm  $N_i = \log T_i$  is nilpotent.

Note that, if  $\mathcal{L}^*$  denotes the local system dual to  $\mathcal{L}$ , then the monodromy operators on the fiber  $L^*$  are given by  $(T_i^*)^{-1}$  and the logarithms are given by  $-N_i^*$ .

**6.2. Some linear algebra.** Suppose  $T$  is an endomorphism of a finite dimensional vector space  $V$  over a field  $F$ . Write  $T^* \in \mathrm{End} V^*$  for the adjoint of  $T$  and let  $(, ) : V \otimes V^* \rightarrow F$  denote the canonical pairing. There is a natural perfect bilinear pairing

$$(, )_T : TV \otimes T^*V^* \rightarrow F, \text{ given by} \\ Tv \otimes T^*\lambda \mapsto (v, T^*\lambda) = (Tv, \lambda).$$

To see that  $(, )_T$  is well-defined, note that  $Tv = 0 \Rightarrow (v, T^*\lambda) = (Tv, \lambda) = 0$ . Similarly,  $T^*\lambda = 0 \Rightarrow (Tv, \lambda) = (v, T^*\lambda) = 0$ . To see that the pairing is perfect, suppose  $(Tv, T^*\lambda)_T = 0$  for all  $\lambda$ . Then  $(Tv, \lambda) = 0$  for all  $\lambda$ . So  $Tv = 0$ . Similarly,  $(Tv, T^*\lambda)_T = 0$  for all  $v$  implies that  $T^*\lambda = 0$ .

**6.3. Koszul and partial Koszul complexes.** The Koszul and partial Koszul complexes are complexes of vector spaces which compute the cohomology groups  $H^p(\mathcal{L}) = H^p(\Delta^{*r}, \mathcal{L})$  and the intersection cohomology groups  $\mathrm{IH}^p(\mathcal{L}) = \mathrm{IH}^p(\Delta^r, \mathcal{L})$  when  $\mathcal{L}$  is as in §6.1. We follow the notation of Kashiwara and Kawai and Cattani, Kaplan and Schmid for these complexes. (See [25, §3.4] and [10].)

Let  $E$  denote an  $r$  dimensional  $\mathbb{F}$ -vector space with basis  $\{e_1, \dots, e_r\}$ . As a  $\mathbb{F}$ -vector space, the Koszul complex is  $K(\mathcal{L}) = L \otimes \wedge^* E$ . It is graded by the usual grading of  $\wedge^* E$ , and it has differential given by

$$d(l \otimes \omega) = \sum_{i=1}^r N_i l \otimes (e_i \wedge \omega). \quad (114)$$

There is a canonical isomorphism  $H^p(\mathcal{L}) = H^p K(\mathcal{L})$ .

Suppose  $J = (j_1, \dots, j_p)$  is a multi-index in  $\{1, \dots, r\}$ . Let  $N_J$  denote the product  $N_{j_1} \cdots N_{j_p}$ , and write  $e_J = e_{j_1} \wedge \cdots \wedge e_{j_p}$ . The partial Koszul complex is the subcomplex  $B(\mathcal{L})$  of  $K(\mathcal{L})$  given by  $\sum_J N_J(L) \otimes e_J$ . By the results of Kashiwara, Kawai [25] and Cattani, Kaplan, Schmid [10], there is a canonical isomorphism  $\mathrm{IH}^p(\mathcal{L}) = H^p(B(\mathcal{L}))$ . In particular, the canonical map  $\mathrm{IH}^p(\Delta^r, \mathcal{L}) \rightarrow H^p(\Delta^{*r}, \mathcal{L})$  is precisely the map induced by the inclusion of complexes  $B(\mathcal{L}) \rightarrow K(\mathcal{L})$ .

Note that a morphism  $f : \mathcal{L} \rightarrow \mathcal{M}$  of local systems on  $\Delta^{*r}$  induces morphisms  $f : K^*(\mathcal{L}) \rightarrow K^*(\mathcal{M})$  and  $f : B^*(\mathcal{L}) \rightarrow B^*(\mathcal{M})$  in the obvious way:  $l \otimes \omega \mapsto f(l) \otimes \omega$ . These induce the corresponding homomorphisms  $H^*(\mathcal{L}) \rightarrow H^*(\mathcal{M})$  and  $\mathrm{IH}^*(\mathcal{L}) \rightarrow \mathrm{IH}^*(\mathcal{M})$ .

Note also that the complexes  $K(\mathcal{L})$  and  $B(\mathcal{L})$  depend on only on the data  $(L, N_1, \dots, N_r)$ .

6.4. If  $(C, d)$  is a complex and  $p$  is a an integer, we write  $Z^p(C) = \{\alpha \in C^p : d\alpha = 0\}$ .

6.5. Suppose  $t \in \mathbb{F}^r$ . Write  $N(t) := \sum_{i=1}^r t_i N_i$ , and write  $\mathcal{L}_t$  for the local system on  $\Delta^*$  with monodromy logarithm  $N(t) \in \mathrm{End} L$ . Then, if  $\mathbb{F}\{e\}$  denotes the free  $\mathbb{F}$ -vector space on one generator  $e$ , we have  $K(\mathcal{L}_t) = L \otimes \wedge^* \mathbb{F}\{e\}$ . As a complex,  $K(\mathcal{L}_t)$  is just the map  $N(t) : L \rightarrow L$  placed in degrees 0 and 1.

The map of vector spaces  $E \rightarrow \mathbb{F}\{e\}$  given by  $e_i \mapsto t_i e$  induces a map  $\lambda_t : \wedge^* E \rightarrow \wedge^* \mathbb{C}\{e\}$ . Write  $\phi_t^\dagger$  for the map  $\mathrm{id}_L \otimes \lambda_t : K(\mathcal{L}) \rightarrow K(\mathcal{L}_t)$ .

If  $\alpha = \sum_{i=1}^r \alpha_i \otimes e_i \in K^1(\mathcal{L})$ , write  $\alpha(t) := \sum_{i=1}^r t_i \alpha_i$ . Then note that  $\phi_t^\dagger \alpha = \alpha(t) \otimes e$ .

**Lemma 115.** *The map  $\phi_t^\dagger : K(\mathcal{L}) \rightarrow K(\mathcal{L}_t)$  is a morphism of complexes.*

*Proof.* Since  $K(\mathcal{L}_t)$  is concentrated in degrees 0 and 1, we just need to check that  $d\phi_t^\dagger(l) = \phi_t^\dagger(dl)$  for  $l \in K^0(\mathcal{L})$ . We compute  $\phi_t^\dagger dl = \phi_t^\dagger \sum N_i l \otimes e_i = \sum t_i N_i l \otimes e = N(t)l \otimes e$  as desired.  $\square$

6.6. Suppose  $t \in \mathbb{Z}_{\geq 0}^r$  and pick a  $a \in \Delta^*$ . Define a test curve  $\phi_t : \Delta^* \rightarrow \Delta^{*r}$  by setting  $\phi_t(s) = a(s^{t_1}, \dots, s^{t_r})$ . Then  $\phi_t$  induces a map

$$\phi_t^* : H^p(\Delta^{*r}, \mathcal{L}) \rightarrow H^p(\Delta^*, \mathcal{L}). \quad (116)$$

We leave it to the reader to check that, under the isomorphism of §6.3,  $\phi_t^* = \phi_t^\dagger$ .

6.7. In the rest of this section,  $\mathcal{H}$  will denote a variation of pure, polarized  $\mathbb{F}$ -Hodge structure of weight  $k$  with unipotent monodromy on the poly-punctured disk  $\Delta^{*r}$ . Write  $H$  for the limit mixed Hodge structure of  $\mathcal{H}$ . In the language of nearby cycles,  $H = \psi_{s_1} \cdots \psi_{s_r} \mathcal{H}$ . As a vector space,  $H$  is isomorphic to the fiber of  $\mathcal{H}$  over any chosen point  $s \in \Delta^{*r}$ . For  $i = 1, \dots, r$ , we write  $T_i$  for the monodromy operator and  $N_i = \log T_i$ . The  $N_i$  then induce morphisms  $N_i : H \rightarrow H(-1)$  of the mixed Hodge structure  $H$ . We write  $H^p(\mathcal{H}) = H^p(\Delta^{*r}, \mathcal{H})$  and  $\mathrm{IH}^p(\mathcal{H}) = \mathrm{IH}^p(\Delta^r, \mathcal{H})$  for the cohomologies of the local system underlying the variation  $\mathcal{H}$ . By [10, 25], these are computed in terms of complexes  $K(\mathcal{H}) = H \otimes \wedge^* E$  and  $B(\mathcal{H})$  as in §6.3. But here the vector space  $E$  is viewed as the pure, weight 2, Hodge structure  $\mathbb{F}(-1)^r$ . The map

$$d : K^p(\mathcal{H}) = H \otimes \wedge^p E \rightarrow K^{p+1}(\mathcal{H}) = H \otimes \wedge^{p+1} E$$

is then a morphism of mixed Hodge structures.

Write  $0 \in \Delta^r$  for the origin. Then we have equalities of Hodge structures  $H^p(\mathcal{H}) = H_0^p(\mathcal{H})$  and  $\mathrm{IH}^p(\mathcal{H}) = \mathrm{IH}_0^p(\mathcal{H})$  where the groups  $H_0^p(\mathcal{H})$  and  $\mathrm{IH}_0^p(\mathcal{H})$  groups defined in 5.2 and 5.3 respectively.

6.8. Write  $F$  for the Hodge filtration on  $H$ . Then the data  $(N_1, \dots, N_r; F)$  defines a nilpotent orbit. This data in turn defines a variation of Hodge structure  $\mathcal{H}_{\mathrm{nilp}}$  of weight  $k$  on  $\Delta_b^{*r}$  for some  $b > 0$  (see (47)). Explicitly,  $\mathcal{H}_{\mathrm{nilp}}$  is the variation with local normal form  $e^{\sum_{i=1}^r z_i N_i} F$  where the  $z_i$  are points in  $\mathbb{C}$  such that  $|e^{2\pi i z_i}| < b$ . We then have equalities of complexes of mixed Hodge structure:  $K(\mathcal{H}) = K(\mathcal{H}_{\mathrm{nilp}})$ ,  $B(\mathcal{H}) = B(\mathcal{H}_{\mathrm{nilp}})$ . In particular,  $H^p(\mathcal{H})$  and  $\mathrm{IH}^p(\mathcal{H})$  depend only on the nilpotent orbit associated to  $\mathcal{H}$ . More precisely:  $\mathcal{H}$  and  $\mathcal{H}_{\mathrm{nilp}}$  restrict to the same local system over  $\Delta_b^{*r}$ , and hence the complexes  $B^\bullet(\mathcal{H})$  and  $B^\bullet(\mathcal{H}_{\mathrm{nilp}})$  are equal. Likewise, all Hodge theoretic data attached to these complexes depends only the limit mixed Hodge structure, which agrees for  $\mathcal{H}$  and  $\mathcal{H}_{\mathrm{nilp}}$  by construction.

We now prove versions of Lemma 112 and Theorem 111 for the normal crossing case.

**Lemma 117.** *Suppose  $r = 1$ . Then  $W_{k+1}H^1(\mathcal{H}) = 0$ .*

*Proof.* The proof is essentially the same as the proof of Lemma 112. Set  $N = N_1$ . As in the proof of Lemma 112, we get an exact sequence of mixed Hodge structures

$$0 \rightarrow \ker N \rightarrow H \xrightarrow{N} H(-1) \rightarrow (\mathrm{coker} N)(-1) \rightarrow 0.$$

Then we use the same reasoning as in Lemma 112 to show that  $W_{k+1}H^1(\mathcal{H}) = 0$ .  $\square$

**Proposition 118.** *Let  $\alpha = \sum_{i=1}^r \alpha_i \otimes e_i \in Z^1(B(\mathcal{H}))$  be a representative of an intersection cohomology class  $\bar{\alpha} \in \mathrm{IH}^1 \mathcal{H}$ . Then, for every  $t \in \mathbb{F}_{\geq 0}^r$ , there is an element  $l(t) \in H$  such that*

$$\alpha(t) = N(t)l(t). \quad (119)$$

*Proof.* Suppose  $t \in \mathbb{F}_{\geq 0}^r$ . Then, if  $F$  denotes the Hodge filtration of  $H$ , the triple  $(H, F, N(t))$  defines a nilpotent orbit in one variable, and, thus, a  $\mathbb{F}$ -variation of Hodge structure on the disk  $\Delta^*$ . Write  $\mathcal{H}_t$  for this variation. Then, by 117,  $W_{k+1}H^1(\mathcal{H}_t) = 0$ .

As in 115, we get a morphism of complexes  $\phi_t^\dagger : K(\mathcal{H}) \rightarrow K(\mathcal{H}_t)$ . Moreover, it is easy to see that  $\phi_t^\dagger$  is, in fact, a morphism of complexes of mixed Hodge structures. By §6.5, the composition  $B^1(\mathcal{H}) \rightarrow K(\mathcal{H}) \rightarrow K(\mathcal{H}_t)$  takes a class  $\alpha \in B^1(\mathcal{H})$  to  $\alpha(t) \otimes e$ .

By the purity theorem of [10],  $W_{k+1} \mathrm{IH}^1(\mathcal{H}) = \mathrm{IH}^1(\mathcal{H})$ . So, it follows from Lemma 117 that the composition  $\mathrm{IH}^1(\mathcal{H}) \rightarrow H^1(\mathcal{H}) \rightarrow H^1(\mathcal{H}_t)$  is zero.

So suppose  $\alpha \in B^1(\mathcal{H})$  represents a class in  $\mathrm{IH}^1(\mathcal{H})$ . Then  $\phi_t^\dagger \alpha = \alpha(t) \otimes e = dl(t) = N(t)l(t)$  for some  $l(t) \in H$ . The result follows.  $\square$

6.9. Suppose  $X \in E^*$ . Then contraction (or interior product) with  $X$  gives a map  $\iota_X : \wedge^p E \rightarrow \wedge^{p-1} E$ . We then get a  $(-1, -1)$  morphism of mixed Hodge structures  $\delta_X : K^p(\mathcal{H}) \rightarrow K^{p-1}(\mathcal{H})$  by setting  $\delta_X(h \otimes \omega) = h \otimes \iota_X(\omega)$ . Note that  $\delta_X$  preserves the restricted Koszul complex  $B(\mathcal{H}) \subset K(\mathcal{H})$ . It is also easy to see that  $\delta_X^2 = 0$ .

Suppose  $t = (t_1, \dots, t_r) \in \mathbb{F}^r$ . Then we write  $X(t) = \sum t_i e_i^* \in E^*$  and  $\delta_t = \delta_{X(t)}$ . We write

$$\Delta_t := d\delta_t + \delta_t d. \quad (120)$$

Note that, for  $\alpha = \sum \alpha_i \otimes e_i \in K^1(\mathcal{H})$ , we have

$$\delta_t \alpha = \sum t_i \alpha_i = \alpha(t). \quad (121)$$

The following lemma is a variant of [10, Lemma 3.7]. (Indeed, it can easily be deduced from [10, Lemma 3.7], but it is also easy enough to give a direct proof.)

**Lemma 122.** For  $t \in \mathbb{F}^r$ , write  $N(t) : K^p(\mathcal{H}) \rightarrow K^p(\mathcal{H})$  for the operator given by  $N(t)(h \otimes \omega) = (N(t)h) \otimes \omega$ . Then  $\Delta_t = N(t)$ .

*Proof.* Suppose  $\alpha = h \otimes \omega \in K^p(\mathcal{H})$  with  $h \in H$  and  $\omega \in \wedge^p E$ . Then

$$\begin{aligned} \Delta_t \alpha &= d\delta_t \alpha + \delta_t d\alpha \\ &= d(h \otimes \iota_{X(t)} \omega) + \delta_t \left( \sum_{i=1}^r N_i h \otimes e_i \wedge \omega \right) \\ &= \sum_{i=1}^r N_i h \otimes e_i \wedge \iota_{X(t)} \omega + \sum_{i=1}^r N_i h \otimes \iota_{X(t)} (e_i \wedge \omega) \\ &= \sum_{i=1}^r N_i h \otimes e_i \wedge \iota_{X(t)} \omega + \sum_{i=1}^r N_i h \otimes t_i \omega - \sum_{i=1}^r N_i h \otimes e_i \wedge \iota_{X(t)} \omega \\ &= \sum_{i=1}^r t_i N_i h \otimes \omega = N(t) \alpha. \end{aligned}$$

□

**6.10. A pairing on the  $B$  complex.** Suppose  $I = (i_1, \dots, i_k)$  is a multi-index in  $\{1, \dots, r\}$ . For  $t \in \mathbb{F}^r$ , set  $t_I = t_{i_1} t_{i_2} \cdots t_{i_k}$ . Using §6.2, we get a pairing

$$\begin{aligned} q_t : B^p(\mathcal{H}) \otimes B^p(\mathcal{H}^*) &\rightarrow \mathbb{F}(-p) \text{ given by} \\ q_t(N_I h \otimes e_I, N_J^* \lambda \otimes e_J) &= \begin{cases} t_I(h, N_J^* \lambda), & I = J; \\ 0, & \text{else.} \end{cases} \end{aligned} \quad (123)$$

Here  $h \in H$  and  $\lambda \in H^*$ , and  $I = (i_1, \dots, i_k)$  and  $J = (j_1, \dots, j_l)$  are multi-indices with  $i_1 < \dots < i_k$  and  $j_1 < \dots < j_l$ .

*Remark 124.* Note that, for  $h \in H$  and  $\mu \in H^*$ ,  $(N_I h, \mu) = (h, N_I^* \mu)$ . It follows that

$$q_t(N_I h \otimes e_I, N_I^* \lambda \otimes e_I) = t_I(N_I h, \lambda).$$

As we mentioned in §6.1, the monodromy logarithms on  $\mathcal{H}^*$  are given by  $-N_i^*$ . So let  $\text{sw} : B^p \mathcal{H} \otimes B^p \mathcal{H}^* \rightarrow B^p \mathcal{H}^* \otimes B^p \mathcal{H}$  be the map switching the factors. From this it follows that  $q_t \circ \text{sw} = (-1)^p q_t$ .

**Proposition 125.** For each  $t \in \mathbb{F}^r$ , the pairing  $q_t$  in (123) is a morphism of mixed Hodge structure. If  $t \in (\mathbb{F}^\times)^r$ , then  $q_t$  is a non-degenerate pairing.

*Proof.* The fact that the pairing is a morphism of Hodge structures follows from the fact that each  $N_i$  induces a morphism  $N_i : H \rightarrow H(-1)$  of mixed Hodge structures (plus a little bit of bookkeeping about the number of Tate twists). The fact that the pairing is non-degenerate for  $t \in (\mathbb{F}^\times)^r$  follows from 6.2. □

**Proposition 126.** For  $\alpha \in B(\mathcal{H})$ ,  $\beta \in B(\mathcal{H}^*)$  and  $t \in \mathbb{F}^r$ , we have

$$q_t(d\alpha, \beta) = q_t(\alpha, \delta_t \beta); \quad (127)$$

$$q_t(\delta_t \alpha, \beta) = q_t(\alpha, d\beta). \quad (128)$$

*Proof.* By the symmetry in the definition (or Remark 124), it suffices to prove (127). For this, suppose  $h \in H$  and  $\lambda \in H^*$ , and set  $\alpha = N_J h \otimes e_J$ ,  $\beta = N_K^* \lambda \otimes e_K$  for multi-indices  $J, K \subset \{1, \dots, r\}$ . Then both sides of (127) are 0 unless there is an element  $i \in K \setminus J$  and an  $\epsilon \in \{\pm 1\}$  such that  $e_i \wedge e_J = \epsilon e_K$ .

Now, we compute

$$\begin{aligned} q_t(d\alpha, \beta) &= q_t(N_i N_j h \otimes e_i \wedge e_j, N_K^* \lambda \otimes e_K) \\ &= \epsilon q_t(N_K h \otimes e_K, N_K^* \lambda \otimes e_K) \\ &= \epsilon t_K(h, N_K^* \lambda). \end{aligned}$$

On the other hand,

$$\begin{aligned} q_t(\alpha, \delta_t \beta) &= q_t(N_j h \otimes e_j, \delta_t N_K^* \lambda \otimes e_K) \\ &= q_t(N_j h \otimes e_j, \delta_t N_K^* \lambda \otimes (\epsilon e_i \wedge e_j)) \\ &= q_t(N_j h \otimes e_j, t_i N_K^* \lambda \otimes (\epsilon e_j)) \\ &= \epsilon t_i q_t(N_j h \otimes e_j, N_K^* \lambda \otimes e_j) \\ &= \epsilon t_i t_j(h, N_K^* \lambda) \\ &= \epsilon t_K(h, N_K^* \lambda). \end{aligned}$$

□

**Lemma 129.** Suppose  $t \in \mathbb{F}_{\geq 0}^r$ . Then, every class in  $\mathrm{IH}^1 \mathcal{H}$  has a representative  $\alpha \in Z^1(B\mathcal{H})$  with  $\delta_t \alpha = 0$ .

*Proof.* Suppose  $\alpha' \in Z^1(B\mathcal{H})$ . Using Proposition 118, find  $l = l(t) \in H$  such that  $N(t)l = \alpha'(t)$ . Then set  $\alpha = \alpha' - dl$ . We have  $\delta_t(\alpha) = \delta_t \alpha' - \delta_t dl = \alpha'(t) - \triangle_t l = \alpha'(t) - N(t)l = 0$ . □

**Corollary 130.** Set  $L_t^p \mathcal{H} = \ker \delta_t \cap \ker d \subset B^p(\mathcal{H})$ . For each  $p$ , the canonical map  $L_t^p \mathcal{H} \rightarrow \mathrm{IH}^p \mathcal{H}$  is a morphism of mixed Hodge structure. The map  $L_t^1 \mathcal{H} \rightarrow \mathrm{IH}^1 \mathcal{H}$  is surjective. In fact, the map  $W_{k+1} L_t^1 \mathcal{H} \rightarrow \mathrm{IH}^1 \mathcal{H}$  is surjective as well.

*Proof.* Since  $\mathrm{IH}^p \mathcal{H} = H^p(B\mathcal{H})$  as a mixed Hodge structure, the map  $L_t^p \mathcal{H} \rightarrow \mathrm{IH}^p \mathcal{H}$  is a morphism of mixed Hodge structure. The second assertion follows from Lemma 129. The last assertion follows from the purity theorem [10, 25] and the strictness of morphisms of mixed Hodge structure with respect to the weight filtration. □

6.11. The restriction of  $q_t$  to  $L_t^1$  gives a pairing

$$q_t : L_t^1(\mathcal{H}) \otimes L_t^1(\mathcal{H}^*) \rightarrow \mathbb{F}(-1). \quad (131)$$

Since  $L_t^1$  is a sub-mixed Hodge structure of  $B^1(\mathcal{H})$ , this is a morphism of mixed Hodge structures. Using the notation that  $\mathcal{H}^\vee = \mathcal{H}^*(1)$ , we get a pairing  $q_t : L_t^1 \mathcal{H} \otimes L_t^1 \mathcal{H}^\vee \rightarrow \mathbb{F}$ .

**Lemma 132.** Suppose  $\alpha \in L_t^1(\mathcal{H}), \lambda \in B^0(\mathcal{H}^*) = \mathcal{H}^*$ . Then  $q_t(\alpha, d\lambda) = 0$ . Similarly, if  $h \in B^0 \mathcal{H}$  and  $\beta \in L_t^1(\mathcal{H}^*)$ , then  $q_t(dh, \beta) = 0$ .

*Proof.* For the first assertion, we have  $q_t(\alpha, d\lambda) = q_t(\delta_t \alpha, \lambda) = 0$ . The second assertion has the same proof (or follows by symmetry). □

**Theorem 133.** Suppose  $t \in \mathbb{F}_{\geq 0}^r$ . The pairing  $q_t : L_t^1 \mathcal{H} \otimes L_t^1 \mathcal{H}^\vee \rightarrow \mathbb{F}$  descends to a pairing

$$h(t) : \mathrm{IH}^1 \mathcal{H} \otimes \mathrm{IH}^1 \mathcal{H}^\vee \rightarrow \mathbb{F}.$$

This pairing, which we call the asymptotic height pairing is a morphism of mixed Hodge structures. Therefore if  $\mathcal{H}$  has weight  $k$ , the pairing factors through a pairing

$$\bar{h}(t) : \mathrm{Gr}_{k+1}^W \mathrm{IH}^1 \mathcal{H} \otimes \mathrm{Gr}_{-k-1}^W \mathrm{IH}^1 \mathcal{H}^\vee \rightarrow \mathbb{F}.$$

If  $c \in \mathbb{F}_{\geq 0}$ , then  $h(ct) = ch(t)$ .



*Proof.* The first assertion follows from Corollary 130 and Lemma 132. The fact that  $h(t)$  is a morphism of Hodge structures follows from the fact that (131) is a morphism of mixed Hodge structures. If  $\mathcal{H}$  is pure of weight  $k$ , then  $\mathrm{IH}^1 \mathcal{H}$  has weights in the interval  $(-\infty, k+1]$ . Similarly,  $\mathrm{IH}^1 \mathcal{H}^\vee$  has weights in the interval  $(-\infty, -k-1]$ . So  $h(t)$  factors through through a pairing  $\bar{h}(t)$  by weight considerations.

To prove the last assertion, note that  $q_{ct} = cq_t$  as a pairing on from  $B^1(\mathcal{H}) \otimes B^1(\mathcal{H}^\vee)$  to  $\mathbb{F}$ . And  $\delta_{ct} = c\delta_t$  on  $B^1(\mathcal{H})$ . It follows that  $L_{ct}^1(\mathcal{H}) = L_t^1(\mathcal{H})$  and similarly  $L_{ct}^1(\mathcal{H}^\vee) = L_c^1(\mathcal{H}^\vee)$ . Therefore  $h(ct) = c(t)$ .  $\square$

**Proposition 134.** *The pairing*

$$h(t) : \mathrm{IH}^1 \mathcal{H} \otimes \mathrm{IH}^1 \mathcal{H}^\vee \rightarrow \mathbb{F} \quad (135)$$

*can be computed as follows. Suppose  $\alpha = \sum_{i=1}^r \alpha_i \otimes e_i \in B^1(\mathcal{H})$  and  $\beta = \sum_{i=1}^r \beta_i \otimes e_i \in B^1(\mathcal{H}^\vee)$  represent intersection cohomology classes  $\bar{\alpha}$  and  $\bar{\beta}$  respectively. Fix  $t$  as above and  $l(t)$  such that  $N(t)l(t) = \alpha(t)$ . Then*

$$h(t)(\alpha, \beta) = \sum_{i=1}^r t_i(\alpha_i - N_i l(t), \beta_i)_{N_i}. \quad (136)$$

*Proof.* Given  $l(t)$  as above, set  $\alpha' = \alpha - dl(t)$ . Then  $\delta_t \alpha' = \alpha(t) - \triangle_t l(t) = 0$ . If we have  $\alpha_i = N_i h_i$  for  $h_i \in H$ , then

$$\begin{aligned} h(t)(\bar{\alpha}, \bar{\beta}) &= q_t(\alpha', \beta) = q_t(\alpha - dl(t), \beta) \\ &= \sum_{i=1}^r t_i(h_i - l(t), \beta) = \sum_{i=1}^r t_i(\alpha_i - N_i l(t), \beta_i)_{N_i} \end{aligned}$$

as desired.  $\square$

*Example 137.* Let  $L = \mathbb{Q}^2$  with basis  $u = (1, 0)$  and  $v = (0, 1)$ , and write  $u^*, v^*$  for the dual basis. Set

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Pick a positive integer  $r$  and let  $N_i = N$  for  $1 \leq i \leq r$ . Let  $\mathcal{L}$  denote the local system on  $\Delta^{*r}$  with monodromy logarithms  $N_i$ . Then  $\mathrm{IH}^1(\mathcal{L}) = \mathbb{Q}^r / \mathbb{Q}$  with the copy of  $\mathbb{Q}$  embedded diagonally. An element of  $\mathrm{IH}^1(\mathcal{L})$  is represented by a sum  $\alpha = \sum_{i=1}^r a_i v \otimes e_i$  with  $a_i \in \mathbb{Q}$ . Similarly, an element of  $\mathrm{IH}^1(\mathcal{L}^*)$  is given by a sum  $\beta = \sum_{i=1}^r b_i u^* \otimes e_i$ . Since  $\mathcal{L}$  underlies a pure variation of Hodge structure of weight  $-1$ ,  $\alpha$  is admissible. We can take

$$l(t) = \frac{\sum_{i=1}^r a_i t_i}{\sum_{i=1}^r t_i} u.$$

Then  $N(t)l(t) = \alpha(t)$ . We claim that

$$h(t)(\alpha, \beta) = \frac{\sum_{i < j} (a_i - a_j)(b_i - b_j)t_i t_j}{\sum t_i}. \quad (138)$$

To see this, we compute

$$\begin{aligned}
h(t)(\alpha, \beta) &= \sum_i t_i(\alpha_i - N_i l(t), \beta_i)_{N_i} \\
&= \sum_i t_i(a_i v - \frac{\sum_j a_j t_j}{\sum_j t_j} N_i u, b_i u^*)_{N_i} \\
&= \sum_i t_i((a_i - \frac{\sum_j a_j t_j}{\sum_j t_j})v, b_i v^*) \\
&= \sum_i t_i(a_i - \frac{\sum_j a_j t_j}{\sum_j t_j})b_i \\
&= \frac{\sum_{i,j} a_i b_i t_i t_j - a_j b_i t_i t_j}{\sum_j t_j} \\
&= \frac{\sum_{i,j} (a_i - a_j) b_i t_i t_j}{\sum_j t_j}
\end{aligned}$$

This is easily manipulated into the form in (138).

## 7. HIGHER DIMENSIONAL PAIRING AND POLARIZATIONS

In this section  $\mathcal{H}$  is a pure variation of  $\mathbb{F}$ -Hodge structure of weight  $k$  with unipotent monodromy on  $\Delta^{*r}$  as in §6.7. We fix a polarization

$$Q : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathbb{F}(-k).$$

This induces a morphism  $a_Q : \mathcal{H} \rightarrow \mathcal{H}^*(-k)$  by  $a_Q(h_1)(h_2) = Q(h_1, h_2)$ . By composing  $h(t)$  with  $a_Q$  we get a pairing on  $\mathrm{IH}^1 \mathcal{H}$  with values in  $\mathbb{F}(-k-1)$ . In fact, we will show that for  $t \in \mathbb{F}_+^r$  and any  $p \in \mathbb{Z}$ , we get a pairing on  $\mathrm{IH}^p \mathcal{H}$ . The goal of this section is to show that these pairings are, in fact, polarizations. We remark that, for  $t = (1, \dots, 1)$ , this is essentially contained in the paper by Cattani, Kaplan and Schmid on Intersection Cohomology [10].

**Definition 139.** For each  $t \in \mathbb{F}^r$  and each  $p \in \mathbb{Z}$ , write

$$R_t^p(\mathcal{H}) = \ker d \cap \ker \Delta_t \cap B^p \mathcal{H}.$$

Note that, for each  $t \in \mathbb{F}^r$ ,  $R_t^p \mathcal{H}$  is a mixed Hodge substructure of  $B^p \mathcal{H}$  containing the Hodge substructure  $L_t^p \mathcal{H}$ .

**7.1. Real splittings.** Let  $\mathcal{D}$  be a classifying space of pure Hodge structures of  $k$  which are polarized by a morphism  $Q : V \otimes V \rightarrow \mathbb{R}(-k)$ . Let  $\theta$  be a nilpotent orbit with values in  $\mathcal{D}$  generated by  $(N_1, \dots, N_r; F)$ . Recall by Theorem (52), the monodromy weight filtration  $W(N)$  is constant on the monodromy cone  $\mathcal{C}$  of positive  $\mathbb{R}$ -linear combinations of  $N_1, \dots, N_r$ . Moreover, if  $W = W(N)[-k]$  then  $(F, W)$  is a mixed Hodge structure for which each element of  $\mathcal{C}$  is a  $(-1, -1)$ -morphism. The associated  $\delta$ -splitting of  $\theta$  is the nilpotent orbit  $\tilde{\theta}$  generated by  $(N_1, \dots, N_r; \tilde{F})$  where  $(\tilde{F}, W) = (e^{-i\delta} F, W)$  is the Deligne  $\delta$ -splitting of  $(F, W)$ .

In particular, since  $\theta$  and  $\tilde{\theta}$  have the same monodromy logarithms  $N_1, \dots, N_r$  they determine the same underlying complex  $B = B(N_1, \dots, N_r)$ . Likewise, since  $\delta$  acts trivially on  $Gr^W$  it follows that  $(F, W)$  and  $(\tilde{F}, W)$  induce the same Hodge structure on  $Gr^W(B)$ . In [10], Cattani, Kaplan and Schmid use this method to reduce questions about the intersection cohomology groups of a general nilpotent orbit to the case where the nilpotent orbit has limit split over  $\mathbb{R}$ .

**Theorem 140.** For each  $p \in \mathbb{Z}$  and each  $t \in \mathbb{F}_+^r$ , the map  $R_t^p \mathcal{H} \rightarrow \mathrm{IH}^p \mathcal{H}$  is onto. Moreover,  $R_t^p \mathcal{H}$  has weights in the interval  $(-\infty, p+k]$ .

*Proof.* It suffices to prove this for real nilpotent orbits. When  $t = (1, 1, \dots, 1)$  and the limit is split over  $\mathbb{R}$ , the first assertion is [10, (3.8)]. We can reduce to the case where the limit is split over  $\mathbb{R}$  as in §7.1. Then we can reduce to the case to  $t = (1, 1, \dots, 1)$  by scaling the  $N_i$  using Lemma (51). The second assertion follows from the definition of the weight filtration  $W$  in terms of  $N(t)$ .  $\square$

**Definition 141.** For each  $t \in \mathbb{F}^r$ , define a pairing  $Q_t$  on  $B^p\mathcal{H}$  by setting

$$Q_t(N_I u \otimes e_I, N_J v \otimes e_J) = \begin{cases} t_I Q(u, N_J v), & I = J; \\ 0, & \text{else.} \end{cases}$$

When  $t = (1, \dots, 1)$ , this pairing is used in [10, (3.4)]. (They write  $S$  instead of  $Q$  for the polarization.)

**Proposition 142.** For each  $t \in (\mathbb{F}^\times)^r$ ,  $Q_t$  defines a pairing

$$Q_t : B^p\mathcal{H} \otimes B^p\mathcal{H} \rightarrow \mathbb{F}(-p-k)$$

which is a non-degenerate morphism of mixed Hodge structure. For  $t \in \mathbb{F}_+^r$ ,  $Q_t$  restricts to a polarization on  $\mathrm{Gr}_{p+k}^W R_t^p \mathcal{H}$ .

*Proof.* Again it suffices to check this for real nilpotent orbits. The non-degeneracy follows from §6.2. The restriction of the pairing to  $R_t^p \mathcal{H}$  then gives a pairing

$$R_t^p \mathcal{H} \otimes R_t^p \mathcal{H} \rightarrow \mathbb{F}(-p-k). \quad (143)$$

Since  $R_t^p \mathcal{H} \subset W_{p+k} B^p \mathcal{H}$ , the pairing (143) vanishes on  $W_{p+k-1} R_t^p \mathcal{H}$  for weight reasons. In the case that  $t = (1, \dots, 1)$  and the limit is split over  $\mathbb{R}$  this pairing is a polarization by [10]. We reduce to that case by applying the  $\delta$ -splitting of §7.1 and rescaling.  $\square$

Note that a polarization on a pure Hodge structure induces a canonical polarization on any sub Hodge structure (by restriction), and it also induces a canonical polarization on any quotient Hodge structure (by orthogonal projection). Proposition 142 therefore gives a canonical polarization

$$\overline{Q}_t : \mathrm{Gr}_{p+k}^W \mathrm{IH}^p \mathcal{H} \otimes \mathrm{Gr}_{p+k}^W \mathrm{IH}^p \mathcal{H} \rightarrow \mathbb{F}(-p-k) \quad (144)$$

for  $t \in \mathbb{F}_+^r$ . Since  $\mathrm{IH}^p \mathcal{H}$  is concentrated in weights  $(-\infty, p+k]$ , the pairing (144) gives a pairing

$$\mathrm{IH}^p \mathcal{H} \otimes \mathrm{IH}^p \mathcal{H} \rightarrow \mathbb{F}(-p-k) \quad (145)$$

also denoted by  $\overline{Q}_t$  (with kernel  $W_{p+k-1} \mathrm{IH}^p \mathcal{H}$ ).

**Lemma 146.** For  $\alpha, \beta \in B(\mathcal{H})$  we have  $Q_t(d\alpha, \beta) = Q_t(\alpha, \delta_t \beta)$ .

*Proof.* The proof is essentially the same as the proof of Proposition 126.  $\square$

**Corollary 147.** Suppose  $\alpha \in L_t^p \mathcal{H}$  and  $\beta \in \ker(R_t^p \mathcal{H} \rightarrow \mathrm{Gr}_{p+k}^W \mathrm{IH}^p \mathcal{H})$ . Then  $Q_t(\alpha, \beta) = 0$ .

*Proof.* Since  $\beta \in R_t^p \mathcal{H}$ , we have  $\beta \in W_{p+k} B^p \mathcal{H}$ . Then, the hypothesis implies that  $\beta = d\gamma + \eta$  where  $\gamma \in W_{p+k} B^{p-1} \mathcal{H}$  and  $\eta \in W_{p+k-1} B^p \mathcal{H}$ . So  $Q_t(\alpha, \beta) = Q_t(\alpha, d\gamma + \eta) = Q_t(\delta_t \alpha, \gamma) + Q_t(\alpha, \eta) = Q_t(\alpha, \eta) = 0$  for weight reasons.  $\square$

7.2. Suppose  $\mathcal{H}$  is pure of weight  $-1$ . Then the map  $a_Q : \mathcal{H} \rightarrow \mathcal{H}^*(1) = \mathcal{H}^\vee$  induces maps  $B^p \mathcal{H} \rightarrow B^p(\mathcal{H}^\vee)$  and  $\mathrm{IH}^p \mathcal{H} \rightarrow \mathrm{IH}^p(\mathcal{H}^\vee)$ , which we also denote by  $a_Q$ . Write  $h_Q(t) : \mathrm{IH}^1 \mathcal{H} \otimes \mathrm{IH}^1 \mathcal{H} \rightarrow \mathbb{Q}$  for the pairing

$$([\alpha], [\beta]) \mapsto h(t)(a_Q[\alpha], [\beta]). \quad (148)$$

**Theorem 149.** Suppose  $[\alpha], [\beta] \in \mathrm{IH}^1 \mathcal{H}$  with  $\mathcal{H}$  pure of weight  $-1$ .

$$h_Q(t)([\alpha], [\beta]) = \overline{Q}_t([\alpha], [\beta]).$$

*Proof.* We can find  $\alpha, \beta \in L_t^1 \mathcal{H}$  representing  $[\alpha]$  and  $[\beta]$  respectively. Then, by Corollary 147 and the definition of  $\overline{Q}_t$ , we have  $h(t)(a_Q[\alpha], [\beta]) = q_t(a_Q \alpha, \beta) = \overline{Q}_t([\alpha], [\beta])$ .  $\square$

The remainder of this section concerns the pairing  $h(t)$  in a special case which will be useful for studying the Ceresa cycle.

**Lemma 150.** Suppose  $L$  is a local system with unipotent monodromy on  $\Delta^{*2}$ . Write  $\gamma_i$ ,  $i = 1, 2$  for the counterclockwise loops around  $s_i = 0$ . (So that  $\pi_1(\Delta^{*2})$  is the free abelian group generated by  $\gamma_1$  and  $\gamma_2$ .) Suppose that the monodromy logarithms  $N_1$  and  $N_2$  are equal. Write  $N = N_1$  and suppose further that  $N^2 = 0$ . Let  $i : \mathbb{Z} \rightarrow \pi_1(\Delta^{*2})$  be given by  $1 \mapsto \gamma_2 - \gamma_1$ . Then

- (i) The map  $NL \rightarrow B^1(L)$  given by  $Nv \mapsto (0, Nv)$  induces an isomorphism  $NL \rightarrow \mathrm{IH}^1(L)$ .
- (ii) The composition

$$NL \rightarrow \mathrm{IH}^1(L) \rightarrow H^1(\Delta^{*2}, L) \xrightarrow{i^*} H^1(\mathbb{Z}, L) = L$$

is the inclusion.

*Proof.* Let  $C(L)$  denote the complex  $\ker N \rightarrow NL$  where the differential is 0. Then under the hypotheses, there is a morphism  $C(L) \rightarrow B(L)$  which is the obvious inclusion on  $C^0$  and where  $C^1(L) \rightarrow B^1(L)$  is given by  $Nv \mapsto (0, Nv)$ . It is easy to see that this morphism is a quasi isomorphism of complexes. This proves (i).

To prove (ii), we note that  $H^1(\mathbb{Z}, L)$  is computed by the Koszul complex  $K(L)$  given by  $L \rightarrow L$  with the 0 differential. Then the composition in (ii) is the map induced on the first cohomology by the following composition of complexes:

$$\begin{array}{ccc} \ker N & \xrightarrow{0} & 0 \oplus NL \\ \downarrow & & \downarrow \\ L & \xrightarrow{d} & L \oplus L \\ \downarrow & & \downarrow (-\mathrm{id}, \mathrm{id}) \\ L & \xrightarrow{0} & L. \end{array}$$

(Here  $d$  denotes the morphism in the Koszul complex.)  $\square$

**Proposition 151.** Suppose  $\mathcal{H}$  is a variation of Hodge structure of weight  $-1$  on  $\Delta^{*2}$  with unipotent monodromy logarithms. Suppose  $N_1 = N_2 = N$  and  $N^2 = 0$ . Let  $Q$  be a polarization of  $\mathcal{H}$ . Identify  $NH$  with  $\mathrm{IH}^1(H)$  using Lemma 150. Then, for each  $t \in \mathbb{Q}_{\geq 0}^2$ , the pairing

$$h_Q(t) : \mathrm{IH}^1 \mathcal{H} \otimes \mathrm{IH}^1 \mathcal{H} \rightarrow \mathbb{Q}$$

amounts to a pairing

$$h_Q(t) : NH \otimes NH \rightarrow \mathbb{Q}.$$

This pairing is given, for  $(t_1, t_2) \neq (0, 0)$  by

$$h_Q(t)(Nh, Nk) = \frac{t_1 t_2}{t_1 + t_2} Q(h, Nk),$$

and vanishes for  $(t_1, t_2) = (0, 0)$ .

*Proof.* Under the isomorphism  $NH \rightarrow \mathrm{IH}^1 \mathcal{H}$  of Lemma 150, the element  $Nh$  (resp.  $Nk$ ) corresponds to the class  $\alpha = Nh \otimes e_2$  (resp.  $\beta = Nk \otimes e_2$ ) in  $B^1 H$ . For,  $t \in \mathbb{Q}_{\geq 0}^2 \setminus \{(0, 0)\}$ , set

$$\begin{aligned} \beta(t) &= Nk \otimes e_2 - d \left( \frac{t_2}{t_1 + t_2} k \right) \\ &= -\frac{t_2}{t_1 + t_2} Nk \otimes e_1 + \frac{t_1}{t_1 + t_2} Nk \otimes e_2. \end{aligned}$$

And set  $\beta(t) = Nk \otimes e_2$  for  $t = (0, 0)$ . Then  $\delta_t \beta = 0$ . So  $\beta(t) \in L_t^1 \mathcal{H}$ . Therefore, the pairing vanishes when  $t = (0, 0)$ , and, when  $t$  is non-zero,  $h_Q(t)(\alpha, \beta) = h(t)(a_Q \alpha, \beta) = q_t(a_Q \alpha, \beta(t)) = t_2 Q(h, \frac{t_1}{t_1 + t_2} Nk) = \frac{t_1 t_2}{t_1 + t_2} Q(h, Nk)$ .  $\square$

## 8. COMPARISON THEOREM

In this section we relate the asymptotic height pairing (133) and the asymptotics of the height of the biextension line bundle. We begin with a minor modification of Remark 18.

**Lemma 152.** *Let  $V_Q = \mathbb{Q}v_0 \oplus U_Q \oplus \mathbb{Q}v_{-2}$  be a finite dimensional vector space. Suppose that  $N$  is a nilpotent endomorphism of  $V_Q$  such that*

$$N(v_0) = \mu v_{-2}, \quad N(U_Q) \subseteq U_Q, \quad N(v_{-2}) = 0.$$

*Let  $e_0 = v_0 + u + cv_{-2}$  be another element of  $V_Q$  such that  $N(e_0) = \mu' v_{-2}$  where  $u \in U_Q$ . Then,  $\mu = \mu'$ .*

*Proof.* By definition,

$$(\mu' - \mu)v_{-2} = N(e_0 - v_0) = N(u + cv_{-2}) = N(u).$$

Since  $N(u) \in U_Q$  and  $U_Q \cap \mathbb{Q}v_{-2} = 0$  it follows that both the right and left hand sides of the previous equation vanish.  $\square$

**Theorem 153.** *Let  $\mathcal{V}$  be an admissible biextension variation with unipotent monodromy over  $\Delta^{*r}$  with underlying normal functions  $v \in \mathrm{ANF}(\Delta^{*r}, \mathcal{H})$  and  $\omega \in \mathrm{ANF}(\Delta^{*r}, \mathcal{H}^\vee)$ . Let  $\mathrm{sing}(v) = \alpha$  and  $\mathrm{sing}(\omega) = \beta$ . Define  $\mu(\mathcal{V})$  as in equation (17). Define  $\mu_1, \dots, \mu_r$  as in Remark (18). Then, for any  $t \in \mathbb{Z}_{\geq 0}^r$ ,*

$$h(t)(\alpha, \beta) = -\mu(\mathcal{V})(t) + \sum_j t_j \mu_j. \quad (154)$$

*In particular,  $h(t)(\alpha, \beta) \equiv -\mu(\mathcal{V})(t)$  modulo a linear function of  $t$ .*

*Proof.* Let  $V$  be the  $\mathbb{Q}$ -vector space corresponding to a reference fiber of  $\mathcal{V}$ . Pick a splitting of the weight filtration  $W$  of  $V$  so that  $V = \mathbb{Q}e_0 \oplus H \oplus \mathbb{Q}e_{-2}$  where:

- (i) the image of  $e_0$  is the canonical generator of  $\mathrm{Gr}_0^W V = \mathbb{Q}$ ;
- (ii)  $H$  maps isomorphically to  $\mathrm{Gr}_{-1}^W$ ;
- (iii)  $e_{-2}$  is the canonical generator of  $W_{-2}V = \mathbb{Q}(1)$ .

Write  $\bar{N}_i$  for the  $i$ -th monodromy operator on  $H$  and write  $N_i$  for the  $i$ -th monodromy operator on  $V$ . Then, in terms of the splitting of  $W$  chosen above, we have

$$N_i = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_i & \bar{N}_i & 0 \\ \gamma_i & -\beta_i & 0 \end{pmatrix}$$

where  $\gamma_i \in \mathbb{Q}(1)$  and  $(\alpha_1, \dots, \alpha_r)$  (resp.  $(\beta_1, \dots, \beta_r)$ ) is a representative of the class  $\alpha$  (resp.  $\beta$ ) in  $Z^1(B(H))$  (resp.  $Z^1(B(H^\vee))$ ). (§6.1 explains why  $\beta_i$  appears with a minus sign in the matrix of  $N_i$ .)

Set  $\bar{N}(t) = \sum t_i \bar{N}_i$ ,  $\gamma(t) = \sum t_i \gamma_i$ , etc. Let  $l(t) \in H$  be an element such that  $\bar{N}(t)l(t) = \alpha(t)$ . Set  $e_0(t) = e_0 - l(t)$ . Then,

$$\begin{aligned} N(t)e_0(t) &= \alpha(t) + \gamma(t)e_{-2} - \bar{N}(t)l(t) + (l(t), \beta(t))e_{-2} \\ &= (\gamma(t) + (l(t), \beta(t)))e_{-2}. \end{aligned}$$

Therefore, by Remark (18) and Lemma (152),  $\mu(\mathcal{V})(t) = \gamma(t) + (l(t), \beta(t))$ . Likewise, we have  $\mu_i = \gamma_i + (l_i, \beta_i)$  where  $\bar{N}_i(l_i) = \alpha_i$ .

On the other hand, by (136)

$$h(t)(\alpha, \beta) = \sum_{i=1}^r t_i (\alpha_i - N_i l(t), \beta_i)_{N_i}.$$

Consequently,

$$\begin{aligned} \mu(\mathcal{V})(t) + h(t)(\alpha, \beta) &= \gamma(t) + (l(t), \beta(t)) + \sum_{i=1}^r t_i (\alpha_i - N_i l(t), \beta_i)_{N_i} \\ &= \gamma(t) + (l(t), \beta(t)) + \sum_{i=1}^r t_i (\alpha_i, \beta_i)_{N_i} - \sum_{i=1}^r t_i (N_i l(t), \beta_i)_{N_i} \\ &= \gamma(t) + (l(t), \beta(t)) + \sum_{i=1}^r t_i (\alpha_i, \beta_i)_{N_i} - \sum_{i=1}^r t_i (l(t), \beta_i) \\ &= \gamma(t) + (l(t), \beta(t)) + \sum_{i=1}^r t_i (\alpha_i, \beta_i)_{N_i} - (l(t), \beta(t)) \\ &= \gamma(t) + \sum_{i=1}^r t_i (\alpha_i, \beta_i)_{N_i} \\ &= \gamma(t) + \sum_{i=1}^r t_i (l_i, \beta_i) \\ &= \sum_{i=1}^r t_i \mu_i \end{aligned}$$

□

## 9. BOUNDARY VALUES

Let  $\mathcal{V} \rightarrow \Delta^{*r}$  be an admissible biextension variation with unipotent monodromy, and  $C = \mathbb{R}_{>0}^r$ . Then, as noted in Remark (18), the function  $\mu(\mathcal{V})(m)$  is defined on the closure  $\bar{C}$  of  $C$ .

**Lemma 155.** *Let  $J \subset \{1, \dots, r\}$ . Then  $\mu(\mathcal{V})(m_1, \dots, m_r)$  is given by a rational function on the subset  $\bar{C}_J \subset \bar{C}$  on which  $m_j > 0$  if  $j \in J$  and  $m_j = 0$  if  $j \notin J$ .*

*Proof.* The proof Theorem (153) shows that  $\mu(\mathcal{V})(m) = \gamma(m) + (\ell(m), \beta(m))$ . Accordingly, the content of the lemma boils down to showing we can find a solution to the equation

$$N(m)\ell(m) = \alpha(m)$$

using rational functions of  $m$ . Without loss of generality, we assume that  $J = \{1, \dots, a\}$ . Let  $\mathcal{H} = Gr_{-1}^W \mathcal{V}$  with reference fiber  $H$ . Then, the nilpotent orbit of  $\mathcal{H}$  determines an associated nilpotent orbit

$$\tilde{\theta}(z) = e^{\sum_k z_k N_k} \tilde{F}$$

with limit mixed Hodge structure split over  $\mathbb{R}$  as in §2. Moreover,  $\tilde{\theta}(z)$  takes values in the appropriate classifying space of pure Hodge structure as soon as  $\text{Im}(z_1), \dots, \text{Im}(z_r) > 0$ . Therefore,

$$\tilde{\theta}_a(z_1, \dots, z_a) = \tilde{\theta}(z_1, \dots, z_a, i, \dots, i)$$

is a nilpotent orbit. By [9], we therefore obtain an  $sl_2$ -triple  $(N(m), H, N^+(m))$  for each  $m \in \bar{C}_J$  where  $H$  is a fixed semisimple element.

To continue, we note that the fact that  $H$  is constant reflects the fact that the monodromy weight filtration  $W_J := W(N(m))$  is constant on  $\bar{C}_J$ . Moreover,  $\alpha(m)$  and  $\beta(m) \in W_{-1}(J)$ . Let  $\alpha_{-1}(m)$  denote the projection of  $\alpha(m)$  to the  $-1$ -eigenspace of  $H$ . Then, since  $N(m) : Gr_1^{W_J} \rightarrow Gr_{-1}^{W_J}$  is an isomorphism, there exists a unique element  $\ell_1(m)$  in the  $+1$  eigenspace of  $H$  such that  $N(m)\ell_1(m) = \alpha_{-1}(m)$ . For weight reasons,

$$(\ell(m), \beta(m)) = (\ell_1(m), \beta(m))$$

Accordingly, the rationality of  $\mu(m)$  boils down to the rationality of the inverse map  $N(m)^{-1} : Gr_{-1}^{W_J} \rightarrow Gr_1^{W_J}$ .  $\square$

Given this lemma, it is natural to ask of if  $\mu(\mathcal{V})$  has a continuous extension from  $C$  to  $\bar{C}$ . To establish this, let  $\mathcal{G}$  be a class of sequences of points in  $C$  with the property that any sequence of points  $\{m(p)\}$  in  $C$  which converges to  $m_o \in \bar{C} - C$  contains a subsequence which belongs to  $\mathcal{G}$ .

**Lemma 156.** *Let  $f : C \rightarrow \mathbb{R}$  be a function which has limit  $L$  along every  $\mathcal{G}$ -sequence which converges to  $m_o \in \bar{C} - C$ . Then,  $f$  has limit  $L$  along every sequence in  $C$  converging to  $m_o$ .*

*Proof.* Let  $m(p)$  be a sequence in  $C$  which converges to  $m_o$ . Then, there exists a real number  $B$  and an index  $p'$  such that  $|f(m(p))| < B$  for all  $p > p'$ . Indeed, if this not true we can find a subsequence  $m(p_i)$  such that  $|f(m(p_i))| > i$  for all  $i$  sufficiently large. By the defining property of  $\mathcal{G}$ , we can find a  $\mathcal{G}$ -subsequence of  $m(p_i)$  which converge to  $m_o$ . By hypothesis,  $f$  has limit  $L$  along  $\mathcal{G}$ -sequences through  $m_o$ .

Suppose now that  $\lim_{p \rightarrow \infty} f(m(p)) \neq L$ . Then, there exists an  $\epsilon > 0$  and a subsequence  $m(p_i)$  such that  $|f(m(p_i)) - L| \geq \epsilon$  for all  $i$ . Passage to a  $\mathcal{G}$ -subsequence of  $m(p_i)$  again produces a contradiction.  $\square$

To apply this lemma, we recall that in [4] the authors defined a notion of  $sl_2$ -sequences on

$$I = \{(z_1, \dots, z_r) \mid \text{Re}(z) \in [0, 1], \quad \text{Im}(z) \in [1, \infty)\}$$

which has the property that every sequence in  $I$  contains an  $sl_2$ -sequence. Given a point  $m_o \in \bar{C} - C$  and a sequence of points  $m(p)$  in  $C$  which converges to  $m_o$ , define

$$\tilde{m}(p) = m(p) / \min(m_1(p), \dots, m_r(p)) \tag{157}$$

keeping in mind that each  $m_j(p) > 0$ . Then,  $\sqrt{-1}\tilde{m}(p)$  is a sequence of points in  $I$ . We say that  $m(p)$  is a  $\mathcal{G}$ -sequence if  $i\tilde{m}(p)$  is a  $sl_2$ -sequence.

As our next preliminary, given a mixed Hodge structure  $(F, W)$  let  $\hat{Y}_{(F, W)}$  denote the Deligne grading of the  $\mathfrak{sl}_2$ -splitting of  $(F, W)$ . Next we note (cf. [4]) that if  $(N_1, \dots, N_r; F, W)$  generate an admissible nilpotent orbit  $\theta$  with limit mixed Hodge structure split over  $\mathbb{R}$  then

$$\hat{Y}_{(e^{icN}F, W)} = \hat{Y}_{(e^{iN}F, W)} \quad (158)$$

for any positive real number  $c$  and any  $N \in \mathcal{C} = \{\sum_j y_j N_j \mid y_1, \dots, y_r > 0\}$ .

Finally, suppose that  $\theta$  is a nilpotent orbit of biextension type with reference fiber  $V$  and let  $e_0 \in V$  project to  $1 \in Gr_0^W$  and  $e_{-2}$  be the generator of  $Gr_{-2}^W$ . Then, for any  $m \in C$

$$\mu(\theta)(m)e_{-2} = \frac{1}{2}[N(m), \hat{Y}_{(e^{iN(m)}F, W)}]e_0. \quad (159)$$

Indeed, by the properties of Deligne systems and the short length of  $W$ , the grading  $\hat{Y}_{(e^{iN}F, W)}$  produces a lift  $v_0$  of  $1 \in Gr_0^W$  such that  $N(v_0)$  belongs to  $W_{-2}$ , so we can apply Lemma (152) to conclude the stated formula.

**Lemma 160.** *Suppose  $f : \bar{C} \rightarrow \mathbb{R}$  is a function, and suppose that, for every sequence  $m(p)$  in  $C$  converging to  $m_0 \in \bar{C}$ ,  $\lim f(m(p)) = f(m_0)$ . Then  $f$  is continuous on  $\bar{C}$ .*

*Proof.* By the definition of continuity,  $f$  is continuous on  $C$ . Suppose that  $m(p)$  is a sequence in  $\bar{C}$  converging to  $m_0$ . For each  $p$ , we can find a sequence  $\{n(p, q)\}_{q=1}^\infty$  of points in  $C$  converging to  $m(p)$ . By our hypothesis, for each  $p$ , we have  $\lim_{q \rightarrow \infty} f(n(p, q)) = f(m(p))$ . So we can find a  $q \in \mathbb{Z}_{>0}$  such that

$$\max(|n(p, q) - m(p)|, |f(n(p, q)) - f(m(p))|) < 2^{-p}$$

where we write  $|n(p, q) - m(p)|$  for the usual Euclidean norm. Set  $n(p) := n(p, q)$ . Then  $n(p) \rightarrow m_0$ , so, by our hypotheses,  $f(n(p)) \rightarrow f(m_0)$ . Moreover,  $\lim_{p \rightarrow \infty} f(n(p)) = \lim_{p \rightarrow \infty} f(m(p))$ . So, since  $m(p)$  was an arbitrary sequence in  $\bar{C}$  converging to  $m_0$ , this proves that  $f$  is continuous.  $\square$

**Theorem 161.**  $\mu(\mathcal{V})$  has a continuous extension from  $C$  to  $\bar{C}$ .

*Proof.* The first step is to observe that  $\mu(\mathcal{V}) = \mu(\theta)$  where  $\theta$  is the nilpotent orbit generated by  $(N_1, \dots, N_r; \hat{F}, M)$  where  $(F, M)$  is  $\mathfrak{sl}_2$ -splitting of the limit mixed Hodge structure of  $\mathcal{V}$ . Indeed, by Remark 18, the value of  $\mu$  depends only on structure of the local monodromy logarithm.

To continue, let  $m(p)$  be a sequence in  $C$  which converges to  $m_o \in \bar{C} - C$  and  $\tilde{m}(p)$  be the corresponding sequence (157) in  $I$ . Then, by property (158) we can rewrite (159) as

$$\mu(\theta)(m(p))e_{-2} = \frac{1}{2}[N(m(p)), \hat{Y}_{(e^{iN(\tilde{m}(p))}F, W)}]e_0$$

By Theorem (2.30) of [4], the grading  $\hat{Y}_{(e^{iN(\tilde{m}(p))}F, W)}$  has limits of the form

$$Y(N(\theta^1), Y(N(\theta^2), \dots, Y_{(F, M)}))$$

along  $\mathfrak{sl}_2$ -sequences (see [4] for notation). The first key thing to know about this limit is that  $N(\theta^1) = N(m_o)$  since  $N(m_o) / \min(m_j(p))$  is the dominant term of  $N(\tilde{m}(p))$ . The second key thing to know is that

$$[N(m_o), Y(N(m_o), Y(N(\theta^2), \dots, Y_{(F, M)}))]$$

belongs to  $W_{-2}\text{End}(V)$ . Therefore,

$$\mu(\theta)(m_o)e_{-2} = \frac{1}{2}[N(m_o), Y(N(m_o), Y(N(\theta^2), \dots, Y_{(F, M)}))]e_0$$

which proves that  $\lim_{m \rightarrow m_o} \mu(\theta)(m) = \mu(\theta)(m_o)$  for sequences  $m(p) \in C$  tending to  $m_o \in \bar{C} - C$ . Now apply Lemma 160.  $\square$



## 10. MIXED EXTENSIONS AND BIEXTENSIONS

In [36, IX.9.3, pp. 421–426], Grothendieck develops the theory of mixed extensions (extensions panachées) in an abelian category. This turns out to be a convenient way to think about mixed extensions of normal functions and the variations of Hodge structure which we have been calling biextension variations. In this section, we briefly review this notion and its relation to the Mumford-Grothendieck's concept of a biextension.

10.1. Suppose  $G$  is a group acting on a set  $X$ . If  $X$  is isomorphic to  $G$  regarded as a  $G$ -set with the action of left multiplication, then  $X$  is called a torsor for  $G$ . If  $X$  is either empty or isomorphic to  $G$ , then  $X$  is called a *pseudo-torsor* for  $G$ .

Similarly a sheaf  $\mathcal{F}$  is called a *pseudo-torsor* for a sheaf of groups  $\mathcal{G}$  if there is an action of  $\mathcal{G}$  on  $\mathcal{F}$  and, for each open  $U$ , either  $\mathcal{F}(U)$  is empty or  $\mathcal{G}(U)$  acts simply transitively on  $\mathcal{F}(U)$ . (See the Stacks Project [37, Tag 03AH].)

10.2. Suppose  $Q_0, Q_1$  and  $Q_2$  are three objects in an abelian category  $\mathbf{C}$ . A *mixed extension* of  $Q_0$  by  $Q_1$  by  $Q_2$  is an object  $X$  of  $\mathbf{C}$  with an increasing filtration  $W_i X$  such that  $W_{-3} X = 0$  and  $W_0 X = X$  together with isomorphisms  $p_i : \text{Gr}_{-i}^W X \xrightarrow{\sim} Q_i$ .

Write  $\text{EXTPAN}(Q_0, Q_1, Q_2)$  for the category of mixed extensions of  $Q_0$  by  $Q_1$  by  $Q_2$ . Here a morphism in  $\text{EXTPAN}(Q_0, Q_1, Q_2)$  from  $X$  to  $X'$  is a morphism of objects in  $\mathbf{C}$  commuting with the isomorphisms from  $p_i$ . Let  $\text{Extpan}(Q_0, Q_1, Q_2)$  denote the set of isomorphism classes in  $\text{EXTPAN}(Q_0, Q_1, Q_2)$ .

From a mixed extension  $X$  of  $Q_0$  by  $Q_1$  by  $Q_2$ , we get an extension  $W_0 X / W_{-2} X$  of  $Q_0$  by  $Q_1$  and an extension  $W_{-1} X$  of  $Q_1$  by  $Q_2$ . In fact, we get a functor

$$\text{EXTPAN}(Q_0, Q_1, Q_2) \rightarrow \text{EXT}(Q_0, Q_1) \times \text{EXT}(Q_1, Q_2). \quad (162)$$

Here  $\text{EXT}(Q_0, Q_1)$  denotes the usual category of extensions of  $Q_0$  by  $Q_1$ . Given extensions  $E_0$  of  $Q_0$  by  $Q_1$  and  $E_1$  of  $Q_1$  by  $Q_2$ , Grothendieck lets  $\text{EXTPAN}(E_0, E_1)$  denote the category of all mixed extensions of  $Q_0$  by  $Q_1$  by  $Q_2$  together with isomorphisms  $\pi_i : W_{-i} X / W_{-i-2} X \xrightarrow{\sim} E_i$  for  $i = 0, 1$ . The morphisms in  $\text{EXTPAN}(E_0, E_1)$  are morphisms  $X \rightarrow X'$  in  $\mathbf{C}$  commuting with the isomorphism  $\pi_i$  for  $i = 0, 1$ .

10.3. **Action of  $\text{Ext}(Q_0, Q_2)$ .** Given an extension  $G$  of  $Q_0$  by  $Q_2$  and an object  $X$  of  $\text{EXTPAN}(E_0, E_1)$ , Grothendieck defines a mixed extension  $G \wedge X$  as follows. First, regard  $X$  as an object of the category  $\text{EXT}(Q_0, E_1)$  of extensions of  $Q_0$  by  $E_1$ . The sequence

$$0 \rightarrow Q_2 \xrightarrow{i} E_1 \rightarrow Q_1 \rightarrow 0$$

gives a functor  $i_*$  from the category  $\text{EXT}(Q_0, Q_2)$  of extensions of  $Q_0$  by  $Q_2$  to  $\text{EXT}(Q_0, E_1)$ . Then using Baer sum in  $\text{EXT}(Q_0, E_1)$  we can define  $G \wedge X$  as  $i_* G + E_1$ . This gives an action of  $\text{EXT}(Q_0, Q_2)$  on  $\text{EXTPAN}(E_0, E_1)$ .

Alternatively we can define  $G \wedge X$  as follows. Suppose  $G$  is given by

$$0 \rightarrow Q_2 \xrightarrow{\iota} G \xrightarrow{\pi} Q_0 \rightarrow 0.$$

Then  $G \wedge X$  is the cohomology of the complex

$$Q_2 \rightarrow G \oplus X \rightarrow Q_0 \quad (163)$$

where the first map is the one sending  $q$  to  $(\iota q, -p_2^{-1}(q))$  and the second map sends  $(g, x)$  to  $\pi(g) - p_0(x)$ .

If we give  $G$  the filtration  $W_i G$  where  $W_{-3} G = 0, W_{-2} G = W_{-1} G = Q_2$  and  $W_0 G = G$ , then this induces a filtration on  $G \oplus X$  making the morphisms in the complex (163) strict (for the obvious filtrations on  $Q_0$  and  $Q_2$ ). We then get a filtration on  $G \wedge X$  making  $G \wedge X$  into an

object in  $\text{EXTPAN}(E_0, E_1)$ . The isomorphism  $\pi_0 : G \wedge X / W_{-2}(G \wedge X) \xrightarrow{\sim} E_0$  is induced from  $\pi_0 : X / W_{-2}X \xrightarrow{\sim} E_0$ . It sends a pair  $(g, x)$  to  $\pi_0 x$ .

A class in  $W_{-1}(G \wedge X)$  can be represented by an element  $(g, x) \in W_{-1}G \oplus W_{-1}X$ . Since  $g \in W_{-1}G$ , we can consider  $g$  as an element in  $Q_2 \subset E_1$ . Then  $\pi_1 : W_{-1}(G \wedge X) \xrightarrow{\sim} E_1$  sends  $(g, x)$  to  $g + \pi_1(x)$ .

It is easy to check that the action  $(G, X) \mapsto G \wedge X$  described above gives a functorial action; that is, a functor  $\text{EXT}(Q_0, Q_2) \times \text{EXTPAN}(E_0, E_1) \rightarrow \text{EXTPAN}(E_0, E_1)$ . The same prescription gives a functor  $\text{EXT}(Q_0, Q_2) \times \text{EXTPAN}(Q_0, Q_1, Q_2) \rightarrow \text{EXTPAN}(Q_0, Q_1, Q_2)$ .

**Proposition 164** (Grothendieck). *Suppose  $Q_i$  are as above for  $i = 0, 1, 2$  and  $E_i$  are as above for  $i = 0, 1$ .*

- (i)  $\text{EXTPAN}(E_0, E_1)$  is a groupoid; that is, every morphism in  $\text{EXTPAN}(E_0, E_1)$  is an isomorphism.
- (ii) The set  $\text{Extpan}(E_0, E_1)$  of isomorphism classes of objects in  $\text{EXTPAN}(E_0, E_1)$  is a pseudo-torsor for  $\text{Ext}(Q_0, Q_2)$ . In other words,  $\text{Extpan}(E_0, E_1)$  is either empty or a torsor for  $\text{Ext}(Q_0, Q_2)$  under the action  $(G, X) \mapsto G \wedge X$ .
- (iii) For  $E \in \text{EXT}(Q_0, Q_1)$  and  $F \in \text{EXT}(Q_1, Q_2)$  as above. Consider the long exact sequence

$$\text{Ext}^1(Q_0, Q_2) \rightarrow \text{Ext}^1(E, Q_2) \rightarrow \text{Ext}^1(Q_1, Q_2) \xrightarrow{\partial} \text{Ext}^2(Q_0, Q_2)$$

arising from the short exact sequence

$$0 \rightarrow Q_1 \rightarrow E \rightarrow Q_0 \rightarrow 0.$$

Write  $\xi \in \text{Ext}^1(Q_1, Q_2)$  for the class of  $F$  and define  $c(E, F) = \partial(\xi)$ . Then  $c(E, F) = 0$  if and only if  $\text{Extpan}(E, F)$  is non-empty.

10.4. There is an obvious map  $\text{Extpan}(Q_0, Q_1, Q_2) \rightarrow \text{Ext}^1(Q_0, Q_1) \times \text{Ext}^1(Q_1, Q_2)$  and if  $(E_0, E_1) \in \text{Ext}^1(Q_0, Q_1) \times \text{Ext}^1(Q_1, Q_2)$  we get a commutative diagram

$$\begin{array}{ccc} \text{Extpan}(E_0, E_1) & \xrightarrow{\quad} & \{(E_0, E_1)\} \\ \downarrow \phi & & \downarrow \\ \text{Extpan}(Q_0, Q_1, Q_2) & \xrightarrow{\pi} & \text{Ext}^1(Q_0, Q_1) \times \text{Ext}^1(Q_1, Q_2). \end{array}$$

We also get an action of  $\text{Hom}(Q_0, Q_1) \times \text{Hom}(Q_1, Q_2)$  on the  $\text{Extpan}(E_0, E_1)$  as follows. Take  $X \in \text{Extpan}(E_0, E_1)$  and  $(f_0, f_1) \in \text{Hom}(Q_0, Q_1) \times \text{Hom}(Q_1, Q_2)$ . Then

$$G(f_0, f_1) := f_1 \cup [E_0] + [E_1] \cup f_0 \in \text{Ext}^1(Q_0, Q_2). \quad (165)$$

Since  $\text{Extpan}(E_0, E_1)$  is a  $\text{Ext}^1(Q_0, Q_2)$ -torsor,  $G(f_0, f_1) \wedge X$  is an element of  $\text{Extpan}(E_0, E_1)$ .

**Proposition 166.** *The map  $\phi$  surjects onto  $\pi^{-1}(E_0, E_1)$ . Moreover, if  $X, Y \in \text{Extpan}(E_0, E_1)$ , then  $\phi(X) = \phi(Y)$  if and only if  $Y = G(f_0, f_1) \wedge X$  for some  $(f_0, f_1) \in \text{Hom}(Q_0, Q_1) \times \text{Hom}(Q_1, Q_2)$ .*

*Proof.* The first statement is obvious. For the second, suppose  $X \in \text{EXTPAN}(E_0, E_1)$  and  $Y = G(f_0, 0) \wedge X$ . The extension  $G := G(f_0, 0)$  is given by the following pullback diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q_2 & \longrightarrow & E_1 & \longrightarrow & Q_1 \longrightarrow 0 \\ & & \parallel & & \uparrow h & & \uparrow f_0 \\ 0 & \longrightarrow & Q_2 & \longrightarrow & G & \longrightarrow & Q_0 \longrightarrow 0 \end{array}$$

of the extension  $E_1$  by the map  $f_0$ . We get a map  $\psi : G \wedge X \rightarrow X$  sending a class represented by  $(g, x)$  to  $x - h(g)$ . This map is well-defined on  $G \wedge X$  because, for  $q \in Q_2$ ,  $\psi(q, -q) = q - h(q) = 0$ . The map  $\psi$  does not, in general, commute with the maps  $\pi_0$  to  $E_0$ : we have  $\pi_0(\psi(g, x)) = \pi_0(x - h(g))$  while  $\pi_0(g, x) = \pi_0(x)$ . However, using the fact that  $h(g) \in W_{-1}X$ , we see that  $\psi$

does commute with  $p_0$ . Then it is easy to see that  $\psi$  commutes with  $p_1$  and  $p_2$ . Thus  $\psi$  induces a morphism in  $\text{EXTPAN}(Q_0, Q_1, Q_2)$  which is easily seen to be an isomorphism. So  $G \wedge X$  is isomorphic to  $X$  in  $\text{EXTPAN}(Q_0, Q_1, Q_2)$ .

The proof that  $G(0, f_1) \wedge X$  is isomorphic to  $X$  in  $\text{EXTPAN}(Q_0, Q_1, Q_2)$  is similar. This proves the “if” part of the proposition.

For the “only if” part, suppose that  $X$  and  $Y$  are objects in  $\text{EXTPAN}(E_0, E_1)$  and  $\psi : Y \rightarrow X$  is an isomorphism between them as objects in  $\text{EXTPAN}(Q_0, Q_1, Q_2)$ . Using  $\pi_i$ , identify  $W_{-i}X/W_{-i-2}X$  and  $W_{-i}Y/W_{-i-2}Y$  with  $E_i$  for  $i = 0, 1$ . Then there exist morphisms  $f_0 : Q_0 \rightarrow Q_1$  and  $f_1 : Q_1 \rightarrow Q_2$  such that  $(W_{-i}/W_{-i-2})(f) = \text{id}_{E_i} + f_i$ . From this it is not hard to see that  $Y \cong G(f_0, f_1) \wedge X$ .  $\square$

**Corollary 167.** *Suppose that  $\text{Hom}(Q_0, Q_1) = \text{Hom}(Q_1, Q_2) = 0$ . Then the fiber of the map*

$$\text{Extpan}(Q_0, Q_1, Q_2) \rightarrow \text{Ext}^1(Q_0, Q_1) \times \text{Ext}^1(Q_1, Q_2)$$

*over  $(E_0, E_1)$  is  $\text{Extpan}(E_0, E_1)$ .*

*Proof.* Obvious.  $\square$

**10.5. Pseudo-biextensions.** Grothendieck’s concept of mixed extension interacts in a rather interesting way with the Mumford-Grothendieck concept of a biextension [36, pp. 156–159]. To explain this fix objects  $Q_0, Q_1$  and  $Q_2$  as above.

Suppose  $E_1, E_2 \in \text{Ext}^1(Q_0, Q_1)$  and  $F_1, F_2 \in \text{Ext}^1(Q_1, Q_2)$ . There are natural composition laws

$$+_1 : \text{Extpan}(E_1, F_1) \times \text{Extpan}(E_2, F_1) \rightarrow \text{Extpan}(E_1 + E_2, F_1)$$

$$+_2 : \text{Extpan}(E_1, F_1) \times \text{Extpan}(E_1, F_2) \rightarrow \text{Extpan}(E_1, F_1 + F_2)$$

satisfying various compatibilities. This is explained in C. Hardouin’s 2005 thesis [21, Theorem 4.4.1]. For the convenience of the reader we explain the operations  $+_1$  and  $+_2$ .

**10.6.** Suppose  $X_i \in \text{Extpan}(E_i, F)$  for  $i = 1, 2$ . Write  $\rho_i : X_i \rightarrow Q_0$  for the composition

$$X_i \rightarrow \text{Gr}_0^W X_i \xrightarrow{p_0} Q_0$$

and write  $j_i : F \rightarrow X_i$  for the canonical injection induced by  $\pi_1^{-1}$ . Then Hardouin defines  $X_1 +_1 X_2$  to be the middle homology group of the complex

$$F \xrightarrow{\begin{pmatrix} j_1 \\ -j_2 \end{pmatrix}} X_1 \oplus X_2 \xrightarrow{\begin{pmatrix} \rho_1 & -\rho_2 \end{pmatrix}} Q_0$$

where we write the arrows in matrix notation. We then get maps

$$\begin{pmatrix} j_1 & 0 \end{pmatrix} : F \rightarrow X_1 +_1 X_2,$$

$$\begin{pmatrix} \rho_1 \\ 0 \end{pmatrix} : X_1 +_1 X_2 \rightarrow Q_0.$$

Combining these maps, we obtain an exact sequence

$$0 \rightarrow F \rightarrow X_1 +_1 X_2 \rightarrow Q_0 \rightarrow 0.$$

And it is not hard to see that this sequence makes  $X_1 +_1 X_2$  into an element of  $\text{Extpan}(E_1 + E_2, F)$  in a canonical way.

10.7. Here is another way to think about the composition  $+_1$ . An object  $X_i \in \text{EXTPAN}(E_i, F)$  can be thought of as an object of  $\text{EXT}^1(Q_0, F)$  whose image in  $\text{EXT}^1(Q_0, Q_1)$  is  $E_i$ . In other words, if we let  $\pi : \text{EXT}^1(Q_0, F) \rightarrow \text{EXT}^1(Q_0, Q_1)$  denote the functor induced by pushforward along the canonical map  $F \rightarrow Q_1$ , then  $\text{EXTPAN}(E_i, F)$  is the fiber category  $\pi^{-1}(E_i)$ . If we take the Baer sum of  $X_1$  and  $X_2$  regarded as objects in  $\text{Ext}^1(Q_0, F)$ , then we clearly get an element of  $\pi^{-1}(E_1 + E_2)$ . It is easy to check that this element is, in fact,  $X_1 +_1 X_2$  as defined above.

10.8. The construction of  $+_2$  is similar, and we explain it in the language of §10.7. Suppose  $X_i$  are objects in  $\text{EXTPAN}(E, F_i)$  for  $i = 1, 2$ . Write  $\iota : \text{EXT}^1(E, Q_2) \rightarrow \text{EXT}^1(Q_1, Q_2)$  for the canonical functor (induced by the inclusion  $Q_1 \rightarrow E$ ). Then we can regard  $X_i$  as object in the fiber category  $\iota^{-1}(F_i)$ . We then define the sum  $X_1 +_2 X_2$  to be the Baer sum of the two  $X_i$  in  $\text{EXT}^1(E, Q_2)$ . It is easy to see that  $X_1 +_2 X_2$  lies in the fiber category  $\iota^{-1}(F_1 + F_2) = \text{EXTPAN}(E, F_1 + F_2)$ .

**Definition 168.** Suppose  $B, C$  and  $A$  are three abelian groups. A *pseudo-biextension* of  $B \times C$  by  $A$  is a set  $E$  equipped with

- (i) an  $A$ -action and a map  $\pi : E \rightarrow B \times C$  giving  $E$  the structure of an  $A$  pseudo-torsor over  $B \times C$ ;
- (ii) commutative operations  $+_1 : E \times_B E \rightarrow E$  and  $+_2 : E \times_C E \rightarrow E$ .

Write  $p_B = \text{pr}_1 \circ \pi$  and  $p_C = \text{pr}_2 \circ \pi$ . For each  $b \in B$ , write  $E_b = p_B^{-1}(b)$  and, for each  $C$ , write  $E_c = p_C^{-1}(c)$ . Then the above data is assumed to satisfy the following properties

- (iii) For each  $b \in B$ , the operation  $+_1$  makes  $E_b$  into an abelian group and the canonical map  $E_b \rightarrow C$  into a group homomorphism. Moreover, the action of  $A$  on  $\ker(E_b \rightarrow C)$  coming from the structure of  $E$  as an  $A$  pseudo-torsor induces a group isomorphism between  $A$  and  $\ker(E_b \rightarrow C)$ .
- (iv) For each  $c \in C$ , the operation  $+_2$  makes  $E_c$  into an abelian group and the canonical map  $E_c \rightarrow B$  into a group homomorphism with kernel equal to  $A$  as in (iii).
- (v) Suppose  $\pi(X_{ij}) = (E_i, F_j)$  for  $i = 1, 2$ . Then

$$(X_{11} +_1 X_{12}) +_2 (X_{21} +_1 X_{22}) = (X_{11} +_2 X_{21}) +_1 (X_{12} +_1 X_{22}).$$

10.9. A surjective pseudo-biextension  $\pi : E \rightarrow B \times C$  is called a *biextension*. Grothendieck studied the notion of biextensions in the category of sheaves of abelian groups in a topos  $T$  in [36]. Suppose  $B, C$  and  $A$  are three sheaves of abelian groups in  $T$ . We define a *pseudo-biextension* of  $B \times C$  by  $A$  to be a sheaf  $E$  in  $T$  equipped with a morphism  $\pi : E \rightarrow B \times C$  along with an  $A$ -action making  $E$  into an  $A$ -torsor over  $B \times C$  and operations  $+_1, +_2$  satisfying the same axioms as in Definition 168 above. A pseudo-biextension  $\pi : E \rightarrow B \times C$  is a *biextension* if  $\pi$  is surjective as a morphism of sheaves.

*Remark 169.* Suppose  $A, B$  and  $C$  are abelian groups, and  $U$  is an extension of  $B \otimes C$  by  $A$ . Write  $E$  for the pull-back of  $U$  to  $B \times C$  via the natural map  $B \times C \rightarrow B \otimes C$  given by  $(b, c) \mapsto b \otimes c$ . The group  $A$  acts on  $U$  via right translation in such a way that  $U$  becomes an  $A$ -torsor over  $B \otimes C$ . Therefore, the pull-back  $E$  of  $U$  becomes an  $A$ -torsor over  $B \times C$ . Write  $p$  for the map  $U \rightarrow B \otimes C$  and  $\pi$  for the map  $E \rightarrow B \times C$ . Suppose  $b_i \in B, c_i \in C$  for  $i = 1, 2$  and  $u_{ij} \in p^{-1}(b_i \otimes c_j)$ . Then  $u_{11} + u_{12} \in p^{-1}(b_1 \otimes C)$ . So we can use the addition in  $U$  to define a map

$$+_1 : E \times_B E \rightarrow E.$$

We can similarly define  $+_2 : E \times_C E \rightarrow E$ . And it is slightly tedious but not hard to show that  $E$  becomes a biextension of  $B \times C$  by  $A$ .

In fact, Grothendieck showed that all biextensions arise in this way provided that  $\mathrm{Tor}_1(B, C) = 0$ . More generally we have an exact sequence

$$0 \rightarrow \mathrm{Ext}^1(B \otimes C, A) \rightarrow \mathrm{Biext}(B \times C, A) \rightarrow \mathrm{Hom}(\mathrm{Tor}_1(B \otimes C), A)$$

where  $\mathrm{Biext}(B \times C, A)$  denotes the set of isomorphisms of biextensions of  $B \times C$  by  $A$ . [36, p.220].

**Theorem 170 (Hardouin).** *Suppose  $\mathbf{C}$  is an abelian category and  $Q, R, P$  are objects satisfying  $\mathrm{Hom}(Q, R) = \mathrm{Hom}(R, P) = 0$ . Then the map*

$$\mathrm{Extpan}(Q, R, P) \rightarrow \mathrm{Ext}^1(Q, R) \times \mathrm{Ext}^1(R, P)$$

*together with the action of  $\mathrm{Ext}^1(Q, P)$  and the operations  $+_1$  and  $+_2$  described above makes  $\mathrm{Extpan}(Q, R, P)$  into a pseudo-biextension of  $\mathrm{Ext}^1(Q, R) \times \mathrm{Ext}^1(R, P)$  by  $\mathrm{Ext}^1(Q, P)$ .*

*Explanation.* The main point is to check that the  $\mathrm{Ext}^1(Q, P)$  action gives  $\mathrm{Extpan}(Q, R, P)$  the structure of a pseudo-torsor. This follows from the condition that  $\mathrm{Hom}(Q, R) = \mathrm{Hom}(R, P) = 0$  and Corollary 167. The rest of the verification is straightforward.  $\square$

**10.10. Sheaf theoretic version.** Suppose now that  $T$  is a topological space and  $\mathcal{C}$  is a stack of abelian categories over  $T$ . So for each open subset  $U$  of  $T$  we have an abelian category  $\mathcal{C}(U)$  and for each inclusion  $i : V \rightarrow U$  we have restriction functor  $i^* : \mathcal{C}(U) \rightarrow \mathcal{C}(V)$ , which we also write as  $X \rightsquigarrow X|_V$ . As in [40, p. 5396], we assume that  $i^*$  is exact. The main example we have in mind is when  $T$  is a complex manifold and  $\mathcal{C}(U)$  is the category of mixed Hodge modules on  $T$ .

For any objects  $A, B \in \mathcal{C}(T)$ , we get a sheaf  $\mathbf{Hom}(A, B)$  sending on open  $U$  to  $\mathrm{Hom}(A|_U, B|_U)$ .

**Lemma 171.** *Suppose that  $A, B$  are objects in  $\mathcal{C}(T)$  satisfying  $\mathbf{Hom}(A, B) = 0$ .*

- (i) *If  $E$  is an extension of  $A$  by  $B$  in  $\mathcal{C}(T)$ , then the identity is the only automorphism of  $E$  as an extension.*
- (ii) *The assignment  $U \rightsquigarrow \mathrm{Ext}^1(A|_U, B|_U)$ , which is naturally a presheaf by the exactness of the restriction functors, is a sheaf.*

*Proof.* (i) If  $\phi$  is an automorphism then  $\phi - \mathrm{id}$  induces an element of  $\mathrm{Hom}(A, B)$  which must be 0.

(ii) Suppose  $\{V_i\}$  is an open cover of  $U$  and, for each  $i$ ,  $E_i \in \mathrm{Ext}^1(A|_{V_i}, B|_{V_i})$  such that, for all  $i, j$ ,  $(E_i)|_{V_i \cap V_j} \cong (E_j)|_{V_i \cap V_j}$ . It follows from (i) that there is a *unique* isomorphism  $\varphi_{ij} : (E_i)|_{V_i \cap V_j} \cong (E_j)|_{V_i \cap V_j}$ . In particular,  $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$  for all  $i, j, k$ . Therefore the  $E_i$  glue together to form an object  $E \in \mathrm{Ext}^1(A|_U, B|_U)$ .  $\square$

We write  $\mathbf{Ext}^1(A, B)$  for the resulting sheaf when  $\mathbf{Hom}(A, B) = 0$ .

**Definition 172.** We call a triple  $Q_\bullet = (Q_0, Q_1, Q_2)$  *disjoint* if  $\mathbf{Hom}(Q_0, Q_1) = \mathbf{Hom}(Q_0, Q_2) = \mathbf{Hom}(Q_1, Q_2) = 0$ .

The main example we have in mind here is a sequence  $Q_i$  of torsion-free variations of pure Hodge structure of weight  $-i$ .

If  $Q_\bullet$  is disjoint then the same argument as above shows that the presheaf

$$U \rightsquigarrow \mathrm{Extpan}(Q_0|_U, Q_1|_U, Q_2|_U)$$

is a sheaf. Similarly, for  $E \in \mathrm{Ext}^1(Q_0, Q_1)$  and  $F \in \mathrm{Ext}^1(Q_1, Q_2)$ , the presheaf  $U \rightsquigarrow \mathrm{Extpan}(E|_U, F|_U)$  is a sheaf. We write  $\mathbf{Extpan}(Q_0, Q_1, Q_2)$  for the first sheaf and  $\mathbf{Extpan}(E, F)$  for the second. By Grothendieck's result,  $\mathbf{Extpan}(E, F)$  is a  $\mathbf{Ext}^1(Q_0, Q_2)$  pseudo-torsor. Moreover,  $\mathbf{Extpan}(E, F)$  is a  $\mathbf{Ext}^1(Q_0, Q_2)$  torsor if the stalks of  $\mathbf{Extpan}(E, F)$  are non-empty.

Using Theorem 170 we see that

$$\mathbf{Extpan}(Q_0, Q_1, Q_2) \rightarrow \mathbf{Ext}^1(Q_0, Q_1) \times \mathbf{Ext}^1(Q_1, Q_2)$$

together with the action of  $\mathbf{Ext}^1(Q_0, Q_2)$  and the operations  $+_1$  and  $+_2$  (which become sheaf morphisms) is a pseudo-biextension of the sheaf  $\mathbf{Ext}^1(Q_0, Q_1) \times \mathbf{Ext}^1(Q_1, Q_2)$  by  $\mathbf{Ext}^1(Q_0, Q_2)$ . It is a biextension if the stalks of  $\mathbf{Extpan}(Q_0, Q_1, Q_2)$  are non-empty.

## 11. MIXED EXTENSIONS OF NORMAL FUNCTIONS

**11.1. The sheaf of mixed extensions.** Suppose  $j : S \rightarrow \bar{S}$  is the inclusion of one complex manifold in another as a Zariski open set. By analogy with §5.1, for an  $A$ -variation of mixed Hodge structure  $\mathcal{H}$  on  $S$  with  $W_{-1}\mathcal{H} = \mathcal{H}$ , write  $\mathbf{NF}(S, \mathcal{H})$  for  $\mathbf{Ext}_{\mathbf{VMHS}(S)}^1(A, \mathcal{H})$  and  $\mathbf{ANF}(S, \mathcal{H})_{\bar{S}}$  for  $\mathbf{Ext}_{\mathbf{VMHS}(S)_{\bar{S}}}^1(A, \mathcal{H})$ . If  $\mathcal{H}$  is torsion free and pure of weight  $-1$ , duality gives an isomorphism  $\mathbf{NF}(S, \mathcal{H}^*(1)) = \mathbf{Ext}_{\mathbf{VMHS}(S)}^1(\mathcal{H}, A)$  and sending  $\mathbf{ANF}(S, \mathcal{H}^*(1))$  to  $\mathbf{Ext}_{\mathbf{VMHS}(S)_{\bar{S}}}^1(\mathcal{H}, A)$ .

Suppose now that  $\mathcal{H}$  is pure of weight  $-1$  and torsion-free. Since the triple  $(A, \mathcal{H}, A(1))$  is disjoint, we get sheaves on  $S$ :  $U \rightsquigarrow \mathbf{NF}(U, \mathcal{H}|_U)$ ,  $U \rightsquigarrow \mathbf{NF}(U, \mathcal{H}|_U^\vee)$  and  $U \rightsquigarrow \mathbf{NF}(U, A(1))$ . We also get a sheaf  $U \rightsquigarrow \mathbf{Extpan}(A, \mathcal{H}|_U, A(1))$ . Write  $\mathbf{NF}(\mathcal{H})$  and  $\mathbf{NF}(A(1))$  for these sheaves on  $S$ . We also get sheaves  $\mathbf{ANF}(\mathcal{H})$  and  $\mathbf{ANF}(A(1))$  on  $\bar{S}$  given by admissible normal functions.

Fix  $\nu \in \mathbf{ANF}(S, \mathcal{H})_{\bar{S}}$  and  $\omega \in \mathbf{ANF}(S, \mathcal{H}^\vee)_{\bar{S}}$ . Then, for any open set  $U \subset S$ , we write  $\mathcal{B}_A(U)$  for the set of isomorphism classes of mixed extensions of  $A$  by  $\mathcal{H}_A$  by  $A(1)$  in  $A - \mathbf{VMHS}(U)$ . Similarly, if  $U \subset \bar{S}$ , we write  $\mathcal{B}_A^{\text{ad}}(U)$  for the set of isomorphism classes of mixed extensions of  $A$  by  $\mathcal{H}_A$  by  $A(1)$  in  $A - \mathbf{VMHS}(U \cap S)_{\bar{U}}^{\text{ad}}$ . The normal functions  $\nu$  and  $\omega$  give rise to extensions  $\nu_A \in \mathbf{Ext}_{A - \mathbf{VMHS}(U \cap S)_{\bar{U}}^{\text{ad}}}^1(A, \mathcal{H}_A)$  and  $\omega_A \in \mathbf{Ext}_{A - \mathbf{VMHS}(U \cap S)_{\bar{U}}^{\text{ad}}}^1(\mathcal{H}_A^\vee, A(1))$ . We write  $\mathcal{B}_A(\nu, \omega)(U)$  (resp.  $\mathcal{B}_A^{\text{ad}}(\nu, \omega)(U)$ ) for set of isomorphism classes of mixed extensions of  $\nu_A$  by  $\omega_A$  in  $A - \mathbf{VMHS}(U)$  (resp.  $A - \mathbf{VMHS}(U \cap S)_{\bar{U}}^{\text{ad}}$ ). When  $A = \mathbb{Z}$  we drop the subscript  $A$  from the notation and simply write  $\mathcal{B}$  (resp.  $\mathcal{B}^{\text{ad}}$ ).

By pullback of variations,  $\mathcal{B}_A$  (resp.  $\mathcal{B}_A^{\text{ad}}$ ) is presheaf on  $S$  (resp.  $\bar{S}$ ).

**Lemma 173.** *The presheaf  $\mathcal{B}_A$  (resp.  $\mathcal{B}_A^{\text{ad}}$ ) is a sheaf on  $S$  (resp.  $\bar{S}$ ).*

*Proof.* This follows from the fact that  $A, \mathcal{H}$  and  $A(1)$  are disjoint in the category of  $A$  variations.  $\square$

**Lemma 174.** *Write  $j_*^{\text{mer}} \mathcal{O}_{\bar{S}}^\times$  for the sheaf of meromorphic functions on  $\bar{S}$  which are regular and non-vanishing on  $S$ . Then*

(i) *The functor*

$$U \rightsquigarrow \mathbf{Ext}_{\mathbf{VMHS}(U)}^1(\mathbb{Z}, \mathbb{Z}(1)) = \mathbf{NF}(U, \mathbb{Z}(1))$$

*defines a sheaf on  $S$  which is canonically isomorphic to  $\mathcal{O}_S^\times$ .*

(ii) *The functor*

$$U \rightsquigarrow \mathbf{Ext}_{\mathbf{VMHS}(U \cap S)_{\bar{U}}^{\text{ad}}}^1(\mathbb{Z}, \mathbb{Z}(1)) = \mathbf{ANF}(U \cap S, \mathbb{Z}(1))_{\bar{U}}$$

*defines a sheaf on  $\bar{S}$  which is canonically isomorphic to  $j_*^{\text{mer}} \mathcal{O}_{\bar{S}}^\times$ .*

*Proof.* Both of these statements are local. The first is well known and follows essentially from the canonical isomorphism  $\mathbf{Ext}_{\mathbf{MHS}}^1(\mathbb{Z}, \mathbb{Z}(1)) = \mathbb{C}^\times$ .

In the case that  $\bar{S} = \Delta^{a+b}$  and  $S = \Delta^{*a} \times \Delta^b$  for non-negative integers  $a$  and  $b$ , the second statement follows from the local normal form of an admissible normal function. Since the statement is local on  $\bar{S}$ , this proves that (ii) holds when  $Y := \bar{S} \setminus S$  is a normal crossing divisor.

For the general case, by using Hironaka, we can find a proper morphism  $p : \bar{S}' \rightarrow \bar{S}$  such that  $p$  is an isomorphism over  $S$  and  $D = p^{-1}Y$  is a normal crossing divisor (with  $Y = \bar{S} \setminus S$  as above). Write  $S' = p^{-1}S$ , and  $j' : S' \rightarrow \bar{S}'$  for the inclusion. Then, if  $U$  is open in  $\bar{S}$ , by M. Saito we have  $\text{ANF}(U \cap S, \mathbb{Z}(1))_U = \text{ANF}(p^{-1}U \cap S', \mathbb{Z}(1))_{p^{-1}U}$ . Since  $j_*^{\text{mer}} \mathcal{O}_S^\times = p_* j'_*{}^{\text{mer}} \mathcal{O}_{S'}^\times$ , this proves that (ii) holds in general.  $\square$

**Corollary 175.** *The sheaf  $\mathcal{B}$  (resp.  $\mathcal{B}^{\text{ad}}$ ) is a biextension (resp. pseudo-biextension) of  $\mathbf{NF}(S, \mathcal{H}) \times \mathbf{NF}(S, \mathcal{H}^\vee)$  by  $\mathcal{O}_S^\times$  (resp. of  $\mathbf{ANF}(S, \mathcal{H})_{\bar{S}} \times \mathbf{ANF}(S, \mathcal{H}^\vee)_{\bar{S}}$  by  $j_*^{\text{mer}} \mathcal{O}_S^\times$ ).*

*Proof.* The fact that  $\mathcal{B}$  and  $\mathcal{B}^{\text{ad}}$  are pseudo-biextensions follows directly from Lemma 174 and Theorem 170. The fact that  $\mathcal{B}$  is a biextension follows from Theorem 79 (applied to the case where  $D = \emptyset$ ).  $\square$

*Remark 176.* If  $\bar{S} \setminus S$  is a normal crossing divisor, then Theorem 79 implies that  $\mathcal{B}^{\text{ad}}$  in Corollary 175 is a biextension. We will prove a stronger result in Theorem 194.

**Corollary 177.** *The sheaf  $\mathcal{B}(v, \omega)$  (resp.  $\mathcal{B}^{\text{ad}}(v, \omega)$ ) is a torsor (resp. pseudo-torsor) for  $\mathcal{O}_S^\times$  (resp.  $j_*^{\text{mer}} \mathcal{O}_S^\times$ ).*

## 12. THE TORSION PAIRING

Suppose  $\mathcal{L}$  is a torsion-free local sytem of  $\mathbb{Z}$  modules on  $\Delta^*$ . Write  $H^1(\mathcal{L})_{\text{tors}}$  for the torsion elements of  $H^1(\mathcal{L}) := H^1(\Delta^*, \mathcal{L})$ . Note that  $H^1(\mathcal{L}^*)$  is canonically isomorphic to  $\text{Ext}_{\Delta^*}^1(\mathcal{L}, \mathbb{Z})$  via the map taking an extension to its dual. Let  $M$  denote the category of mixed extensions of  $\mathbb{Z}$  by  $\mathcal{L}$  by  $\mathbb{Z}$  (in the category of sheaves of abelian groups over  $\Delta^*$ ). We call a mixed extension  $X$  in  $M$  *restricted* if  $X/W_{-2}X \in H^1(\mathcal{L})_{\text{tors}}$  and  $W_{-1}X \in H^1(\mathcal{L}^*)_{\text{tors}}$ . Write  $\mathcal{R}$  for the set of isomorphism classes of such mixed extensions. It comes equipped with an obvious map  $\mathcal{R} \rightarrow H^1(\mathcal{L})_{\text{tors}} \times H^1(\mathcal{L}^*)_{\text{tors}}$ .

**Proposition 178.** *We have an action of  $\mathbb{Z} = \text{Ext}_{\Delta^*}^1(\mathbb{Z}, \mathbb{Z})$  on  $\mathcal{R}$  along with operations  $+_1$  and  $+_2$  on  $\mathcal{R}$ . These make  $\mathcal{R}$  into a biextension of  $H^1(\mathcal{L})_{\text{tors}} \times H^1(\mathcal{L}^*)_{\text{tors}}$  by  $\mathbb{Z}$ .*

*Proof.* Suppose  $E_0$  is an extension of  $\mathbb{Z}$  by  $\mathcal{L}$  and  $E_1$  is an extension of  $\mathcal{L}$  by  $\mathbb{Z}$ . If  $E_0$  and  $E_1$  have torsion cohomology classes, then, for  $(f_0, f_1) \in \text{Hom}(\mathbb{Z}, \mathcal{L}) \times \text{Hom}(\mathcal{L}, \mathbb{Z})$ , the class  $G(f_0, f_1) \in \text{Ext}_{\Delta^*}^1(\mathbb{Z}, \mathbb{Z}) = H^1(\Delta^*, \mathbb{Z})$  (given by (165)) is trivial. It follows by Proposition 166 that  $\text{Extpan}(E_0, E_1)$  injects into  $\text{Extpan}(\mathbb{Z}, \mathcal{L}, \mathbb{Z})$  (where the  $\text{Extpan}$  sets are taken with respect to the category of sheaves of abelian groups on  $\Delta^*$ ). The set  $\text{Extpan}(E_0, E_1)$  is an  $\text{Ext}_{\Delta^*}^1(\mathbb{Z}, \mathbb{Z})$ -torsor. In other words, it is a  $\mathbb{Z}$ -torsor. Consequently,  $\mathcal{R}$  has an action of  $\mathbb{Z}$  making it into a  $\mathbb{Z}$ -torsor over  $H^1(\mathcal{L})_{\text{tors}} \times H^1(\mathcal{L}^*)_{\text{tors}}$ .

The operations  $+_1$  and  $+_2$  are defined as in §10.5. The rest of the verification is left to the reader.  $\square$

By the results in [36] summarized in Remark 169, biextensions of  $H^1(\mathcal{L})_{\text{tors}} \otimes H^1(\mathcal{L}^*)_{\text{tors}}$  by  $\mathbb{Z}$  are classified by the group  $\text{Ext}^1(H^1(\mathcal{L})_{\text{tors}} \otimes H^1(\mathcal{L}^*)_{\text{tors}}, \mathbb{Z})$ . This group sits in an exact sequence

$$\begin{aligned} \text{Hom}(H^1(\mathcal{L})_{\text{tors}} \otimes H^1(\mathcal{L}^*)_{\text{tors}}, \mathbb{Q}) &\rightarrow \text{Hom}(H^1(\mathcal{L})_{\text{tors}} \otimes H^1(\mathcal{L}^*)_{\text{tors}}, \mathbb{Q}/\mathbb{Z}) \\ &\rightarrow \text{Ext}^1(H^1(\mathcal{L})_{\text{tors}} \otimes H^1(\mathcal{L}^*)_{\text{tors}}, \mathbb{Z}) \rightarrow \text{Ext}(H^1(\mathcal{L})_{\text{tors}} \otimes H^1(\mathcal{L}^*)_{\text{tors}}, \mathbb{Q}). \end{aligned}$$

Since the first and last groups are 0, we have an isomorphism between the second and third groups. Thus, the biextension gives rise to a bilinear pairing

$$\tau : H^1(\mathcal{L})_{\text{tors}} \otimes H^1(\mathcal{L}^*)_{\text{tors}} \rightarrow \mathbb{Q}/\mathbb{Z}$$

which we call the *Grothendieck torsion pairing* or just the *torsion pairing*.

We want to compute the torsion pairing in an explicit way. To do this, write  $\mathcal{R}_Q$  for the set of isomorphism classes of mixed extensions  $X$  of  $Q$  by  $\mathcal{L}_Q$  by  $Q$  which are restricted in the sense that  $X/W_{-2}$  and  $W_{-1}X$  are both 0 (in  $H^1(\mathcal{L}_Q)$  and  $H^1(\mathcal{L}_Q^*)$  respectively). The same argument as above shows that  $\mathcal{R}_Q$  has the structure of a biextension, but this time it is a biextension over the trivial group 0. Therefore, there is a canonical isomorphism

$$\mathcal{R}_Q = \text{Ext}_{\Delta^*}^1(Q, Q) = Q. \quad (179)$$

Moreover, tensoring with  $Q$  gives a morphism of biextensions  $X \rightsquigarrow X_Q$  from  $\mathcal{R}$  to  $\mathcal{R}_Q$ . (See [36, p. 162] for the notion of a morphism of biextensions.) So, for  $X \in \mathcal{R}$ , we get a rational number  $\tilde{\tau}X$  given by the image of  $X_Q$  under (179).

**Definition 180.** Write  $\mathcal{R}'$  for the set of triples  $(\alpha, \beta, \gamma) \in H^1(\mathcal{L}_{\text{tors}}) \times H^1(\mathcal{L}^*)_{\text{tors}} \times Q$  such that there exists  $X \in \mathcal{R}$  with  $\pi X = (\alpha, \beta)$ ,  $\tilde{\tau}X = \gamma$ .

Since  $X \rightarrow X_Q$  is a morphism of biextensions from  $\mathcal{R}$  to  $\mathcal{R}_Q$ , we have, for  $i = 1, 2$ ,

$$\tilde{\tau}(X +_i X') = \tilde{\tau}(X) + \tilde{\tau}(X')$$

whenever  $+_i$  is defined. Moreover, for  $n \in \mathbb{Z}$ ,  $\tilde{\tau}(n + X) = n + \tilde{\tau}X$ .

From this, it is not hard to see that the group  $\mathbb{Z}$  acts on  $\mathcal{R}'$  by the rule  $n + (\alpha, \beta, \gamma) = (\alpha, \beta, n + \gamma)$ . Moreover, by setting

$$\begin{aligned} (\alpha, \beta, \gamma) +_1 (\alpha', \beta, \gamma') &= (\alpha + \alpha', \beta, \gamma + \gamma') \\ (\alpha, \beta, \gamma) +_2 (\alpha, \beta', \gamma') &= (\alpha, \beta + \beta', \gamma + \gamma') \end{aligned}$$

we get the structure of a biextension on  $\mathcal{R}'$ . In fact, we get an isomorphism of biextensions

$$\phi : \mathcal{R} \rightarrow \mathcal{R}' \quad (181)$$

given by  $X \mapsto (X/W_{-2}X, W_{-1}X, \tilde{\tau}X)$ . All this leads to the following proposition.

**Proposition 182.** Suppose  $(\alpha, \beta) = \pi X$  for  $X \in \mathcal{R}$ . Then  $\tau(\alpha \otimes \beta) = \tilde{\tau}X \pmod{\mathbb{Z}}$ .

*Proof.* Write  $t$  for the reduction of  $\tilde{\tau}$  modulo  $\mathbb{Z}$ . Then  $t$  gives a map

$$t : H^1(\mathcal{L})_{\text{tors}} \otimes H^1(\mathcal{L}^*)_{\text{tors}} \rightarrow Q/\mathbb{Z}, \quad (183)$$

and what we have to prove is that  $\tau = t$ . Using  $t$  and the isomorphism

$$\text{Hom}(H^1(\mathcal{L})_{\text{tors}} \otimes H^1(\mathcal{L}^*)_{\text{tors}}, Q/\mathbb{Z}) \rightarrow \text{Ext}^1(H^1(\mathcal{L})_{\text{tors}} \otimes H^1(\mathcal{L}^*)_{\text{tors}}, \mathbb{Z}), \quad (184)$$

we see that  $t$  corresponds to the extension of  $H^1(\mathcal{L})_{\text{tors}} \otimes H^1(\mathcal{L}^*)_{\text{tors}}$  by  $\mathbb{Z}$  whose fiber over  $\alpha \otimes \beta$  is the set of triples  $(\alpha, \beta, \gamma)$  with  $\gamma \in t(\alpha \otimes \beta)$  (where we view  $t(\alpha \otimes \beta)$  as a coset of  $\mathbb{Z}$  in  $Q$ ). But this is exactly the fiber of  $\mathcal{R}'$  over  $\alpha \otimes \beta$ . And this shows that the  $\mathcal{R}'$  is the image of  $t$  under the map

$$\text{Ext}^1(H^1(\mathcal{L})_{\text{tors}} \otimes H^1(\mathcal{L}^*)_{\text{tors}}, \mathbb{Z}) \rightarrow \text{Biext}(H^1(\mathcal{L})_{\text{tors}} \times H^1(\mathcal{L}^*)_{\text{tors}}, \mathbb{Z}). \quad (185)$$

Since  $\mathcal{R}$  is isomorphic to  $\mathcal{R}'$ , this shows that  $\tau = t$ .  $\square$

To write the torsion pairing explicitly, let  $L$  denote the fiber of  $\mathcal{L}$  at a chosen point  $s_0 \in \Delta^*$  and write  $T \in \text{Aut } L$  for the monodromy operator. Then  $H^1(\mathcal{L})$  is computed by the Koszul complex  $K_{\mathbb{Z}}(L)$  given by

$$L \xrightarrow{T-1} L$$

in degrees 0 and 1. So that  $H^0\mathcal{L} = L^T$  and  $H^1\mathcal{L} = L/(T-1)L$ . We can make the computation of  $H^1\mathcal{L}$  explicit if we view an element of  $H^1\mathcal{L}$  as an extension

$$0 \rightarrow L \rightarrow E \xrightarrow{p} \mathbb{Z} \rightarrow 0 \quad (186)$$



in the category of abelian groups equipped with a  $\mathbb{Z}$ -action (where the action on  $\mathbb{Z}$  on itself is taken to be trivial).

If we pick an element  $e \in E$  such that  $p(e) = 1$ , then  $(T-1)e \in L$ . We can change  $e$  to  $e' := e + l$  for some  $l \in L$ . Then  $(T-1)e' = (T-1)e + (T-1)l$ . Therefore the class  $[(T-1)e] \in L/(T-1)L$  is well defined. And it is easy to see that this identifies  $H^1(\Delta^*, \mathcal{L})$  with  $L/(T-1)L$ .

**Corollary 187.** *Under the above identification, we have*

$$H^1(\mathcal{L})_{\text{tors}} = \frac{L \cap (T-1)L_{\mathbb{Q}}}{(T-1)L}.$$

*Proof.* Obvious. □

Suppose  $X \in \mathcal{R}$  with  $\pi(X) = (\alpha, \beta) \in H^1(\mathcal{L})_{\text{tors}} \times H^1(\mathcal{L}^*)_{\text{tors}}$ . Write  $V$  for the fiber of  $X$  at  $s_0$ , and write  $\tilde{T} \in \text{Aut } V$  for the monodromy action. This action preserves the filtration  $W$  on  $V$ .

**Proposition 188.** *Suppose  $X \in \mathcal{R}$  has monodromy matrix  $\tilde{T}$ . There exists an  $e_0 \in V_{\mathbb{Q}}$  with the following properties*

- (i) *the projection of  $e_0$  to  $\text{Gr}^W V_{\mathbb{Q}}$  is equal to 1 under the identification  $p_0 : \text{Gr}^W_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}$ ;*
- (ii)  *$(\tilde{T} - 1)e_0 \in W_{-2}V_{\mathbb{Q}}$ .*

*For any such element  $e_0$ , we have  $\tilde{\tau}(X) = p_{-2}((\tilde{T} - 1)(e_0))$ .*

*Proof.* When tensored with  $\mathbb{Q}$ , the extensions  $X/W_{-2}X$  and  $W_{-1}X$  in  $H^1(\mathcal{L})$  respectively  $H^1(\mathcal{L}^*)$  become trivial. The existence of  $e_0 \in V_{\mathbb{Q}}$  follows from the triviality of  $X_{\mathbb{Q}}/W_{-2}X_{\mathbb{Q}}$ . Then the equality  $\tilde{\tau}X = p_{-2}((\tilde{T} - 1)(e_0))$  follows from the definition of  $\tilde{\tau}$  as the image of  $X_{\mathbb{Q}}$  in  $\text{Ext}^1_{\Delta^*}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}$ . □

**Proposition 189.** *Suppose  $\alpha \in L \cap (T-1)L_{\mathbb{Q}}$  represents a class in  $H^1(\mathcal{L})_{\text{tors}}$  and  $\beta \in L^*$  represents a class in  $H^1(\mathcal{L}^*)_{\text{tors}}$ . Pick  $\ell \in L_{\mathbb{Q}}$  such that  $\alpha = (T-1)\ell$ . Then*

$$\tau([\alpha], [\beta]) = -(\ell, \beta) \pmod{\mathbb{Z}}.$$

*Proof.* We can represent a mixed extension  $X \in p^{-1}(\alpha, \beta)$  by giving the monodromy with respect to a basis consisting of

- (i) an element  $e_0 \in X$  projecting to 1 under the isomorphism  $\text{Gr}_0^W X = \mathbb{Z}$ ,
- (ii) elements in  $W_{-1}X$  lifting a basis of  $L$  under the map  $\text{Gr}_{-1}^W X = L$ ,
- (iii) the generator  $e_{-2}$  of  $W_{-2}X = \mathbb{Z}$ .

In matrix form, we then have

$$\tilde{T} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & T & 0 \\ m & \beta & 1 \end{pmatrix}$$

for some  $m \in \mathbb{Z}$ . Over  $\mathbb{Q}$  we can change  $e_0$  to  $e'_0 = e_0 - \ell$ . Then  $\tilde{T}e'_0 = e_0 + \alpha + me_{-2} - T\ell - (\ell, \beta)e_{-2} = e'_0 + \alpha + me_{-2} - (T-1)\ell - (\ell, \beta)e_{-2} = e'_0 + (m - (\ell, \beta))e_{-2}$ . So if we change the basis by changing  $e_0$  to  $e'_0$  the matrix for  $\tilde{T}$  becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & T & 0 \\ m - (\ell, \beta) & \beta & 1 \end{pmatrix}.$$

So  $\tilde{\tau}(X) = m - (\ell, \beta)$ , and  $\tau(X) = -(\ell, \beta)$  as desired. □

Suppose now that  $\mathcal{H}$  is a torsion free weight  $-1$  variation of Hodge structure on  $\Delta^*$ . Fix admissible normal functions  $\nu \in \text{ANF}(\Delta^*, \mathcal{H})_\Delta$  and  $\omega \in \text{ANF}(\Delta^*, \mathcal{H}^\vee)_\Delta$ . A biextension variation of Hodge structure  $\mathcal{V} \in \mathcal{B}^{\text{ad}}(\nu, \omega)$  then gives a mixed extension in the category of admissible variations. The underlying local system  $\mathcal{V}_\mathbb{Z}$  is then a restricted mixed extension in  $\mathcal{R}$ .

**Corollary 190.** *Suppose  $\mathcal{H}$  has unipotent monodromy. Then  $\mu(\mathcal{V}) = \tilde{\tau}\mathcal{V}_\mathbb{Z}$ .*

*Proof.* This follows from Proposition 188.  $\square$

### 13. EXTENSION THEOREM

**13.1. Setup and torsion pairings of smooth divisors.** We take  $\bar{S}$  to be a complex manifold,  $j : S \rightarrow \bar{S}$  the inclusion of a Zariski open and  $\mathcal{H}$  to be a weight  $-1$  variation of pure Hodge structure without torsion on  $S$ . We write  $Y := \bar{S} \setminus S$ ,  $Y^{(1)} = \{Y_1, \dots, Y_k\}$  for the set of components of  $Y$  which are codimension 1 in  $\bar{S}$ . We write  $Y^{(2)}$  for the union of the codimension 2 components of  $Y$  and the singular locus of  $Y$ . We write  $S' := \bar{S} \setminus Y^{(2)}$ , and  $Y' = Y \cap S'$ . Then  $Y'$  is a smooth divisor in  $S'$  with components  $Y'_i := Y_i \cap S'$ .

**Definition 191.** Suppose  $\mathcal{L}$  is a  $\mathbb{Z}$  local system on  $S$ . Set

$$\text{IH}_\mathbb{Z}^1 \mathcal{L} := \{\alpha \in H^1(S, \mathcal{L}) : \alpha_\mathbb{Q} \in \text{IH}^1(S, \mathcal{L}_\mathbb{Q})\}.$$

By Theorem 108, the map  $\text{cl} : \text{ANF}(S, \mathcal{H})_{\bar{S}} \rightarrow H^1(S, \mathcal{H})$  factors through  $\text{IH}_\mathbb{Z}^1 \mathcal{H}$ .

For each  $Y_i \in Y^{(1)}$  we are going to define a pairing

$$\tau_i : \text{IH}_\mathbb{Z}^1 \mathcal{H} \otimes \text{IH}_\mathbb{Z}^1 \mathcal{H}^\vee \rightarrow \mathbb{Q}/\mathbb{Z} \quad (192)$$

which we will call the  $i$ -th torsion pairing.

We define (192) first in the case that  $\bar{S} = \Delta$ ,  $S = \Delta^*$  and  $Y = \{0\}$ . Note that, in this case,  $\text{IH}_\mathbb{Z}^1(\Delta, \mathcal{H}) = \{\alpha \in H^1(\Delta^*, \mathcal{H}) : \alpha_\mathbb{Q} = 0\}$ . In other words, it is just the set of torsion elements of  $H^1(\Delta^*, \mathcal{H})$ . So, in this case the pairing in (192) is just given by the torsion pairing  $\tau$ .

In general we take a test curve  $\bar{\varphi} : \Delta \rightarrow S'$  intersecting  $Y_i \in Y^{(1)}$  transversally at a point  $\bar{\varphi}(0)$ . By Theorem 111, the pull back of  $\text{IH}_\mathbb{Z}^1 \mathcal{H}$  to  $\Delta^*$  consists of torsion classes. Then we can define  $\tau_i$  to be the torsion pairing restricted to the test curve. It is easy to see that this definition does not depend on the test curve.

### 13.2. Extension of the line bundle $\mathcal{L}(\nu, \omega)$ .

**Lemma 193.** *Suppose  $j : S \rightarrow \bar{S}$  is as in §13.1. Suppose that  $\text{codim}_{\bar{S}} Y \geq 2$ . Let  $\pi : \bar{T} \rightarrow \bar{S}$  be a proper morphism from a complex manifold,  $\bar{T}$ , which is an isomorphism over  $S$ , and suppose  $\mathcal{M}$  is a line bundle on  $\bar{T}$ . Set  $\mathcal{L} := (\pi_* \mathcal{M})^{\vee\vee}$ , the reflexive hull of  $\pi_* \mathcal{M}$ . Then  $\mathcal{L}$  is a line bundle on  $\bar{S}$  and there is a canonical isomorphism  $\mathcal{L}|_S \cong \mathcal{M}|_S$ .*

*Proof.* Write  $i : S \rightarrow \bar{T}$  for the open immersion lifting  $j$ . Since  $\pi$  is proper,  $\pi_* \mathcal{M}$  is coherent. We have  $j^* \pi_* \mathcal{M} = i^* \mathcal{M}$  by smooth base change. So  $j^* \mathcal{L} = j^*(\pi_* (\mathcal{M})^{\vee\vee}) = (i^* \mathcal{M})^{\vee\vee} = i^* \mathcal{M}$ .

Now,  $\mathcal{L}$  is reflexive and agrees with a line bundle outside of codimension 2. It is therefore a rank 1 reflexive sheaf. Therefore, since  $\bar{S}$  is a complex manifold,  $\mathcal{L}$  is a line bundle. (This is in [29].)  $\square$

**Theorem 194.** *Suppose  $S, \bar{S}, Y$  and  $\mathcal{H}$  are as in the beginning of §13.1, so that  $\mathcal{H}$  is a torsion free variation of Hodge structure on  $S$  of weight  $-1$  and  $\bar{S}$  is a smooth partial compactification of  $S$ . Suppose*

$$(\nu, \omega) \in \text{ANF}(S, \mathcal{H})_{\bar{S}} \times \text{ANF}(S, \mathcal{H}^\vee)_{\bar{S}}.$$

*Then each choice of coset representative  $\tilde{\tau}_i$  of  $\tau_i := \tau_i(\text{cl } \nu, \text{cl } \omega) \in \mathbb{Q}/\mathbb{Z}$  determines a unique extension  $\bar{\mathcal{L}}$  of  $\mathcal{L} := \mathcal{L}(\nu, \omega)$  to  $\bar{S}$  whose non-vanishing sections are admissible biextension variations  $\mathcal{V}$  with the following property: for every test curve  $\bar{\varphi} : \Delta \rightarrow S'$  intersecting  $Y_i$  transversally at  $\bar{\varphi}(0)$ ,  $\mu(\bar{\varphi}^* \mathcal{V}) = \tilde{\tau}_i$ .*

*Proof.* First assume that  $Y$  is a normal crossing divisor. Then, by Corollary 175, the sheaf  $\mathcal{B}^{\text{ad}}(\nu, \omega)$  of admissible biextension variations is a pseudo-torsor for the group of non-vanishing meromorphic functions with poles along  $Y$ . By Theorem 79, this pseudo-torsor is actually a torsor for the sheaf of non-vanishing meromorphic functions. For every section  $\mathcal{V}$  of  $\mathcal{B}^{\text{ad}}(\nu, \omega)$  and every test curve  $\bar{\varphi} : \Delta \rightarrow S'$  intersecting the divisor  $Y$  transversally at  $y = \bar{\varphi}(0)$  with  $y$  a point in  $Y_i$ ,  $\mu(\varphi^* \mathcal{V}) = \tilde{\tau} \varphi^* \mathcal{V}$  is a coset representative of  $\tau_i$ . Fixing the  $\tilde{\tau}_i$  then reduces the torsor to a  $\mathcal{O}_S^\times$  torsor  $\bar{\mathcal{B}}^{\text{ad}}(\nu, \omega)$ . Equivalently, it gives a line bundle  $\bar{\mathcal{L}}$  as desired.

Now, in the general case, set  $Y' = Y \cap S'$ . Then, as  $Y'$  is a normal crossing divisor in  $S'$ , there is an extension  $\mathcal{L}'$  of  $\mathcal{L}$  to  $S'$ . Now, use Hironaka to find a proper morphism  $\pi : \bar{T} \rightarrow \bar{S}$  which is an isomorphism over  $S'$  such that the inverse image of  $Y$  under  $\pi$  is a normal crossing divisor  $D$ . Write  $D = \cup_{i=1}^m D_i$  in such a way that the  $D_i$  ( $i = 1, \dots, k$ ) are the strict transforms of the  $Y_i$ . Pick rational numbers  $\tilde{\tau}_i$  lifting the  $\tau_i$  keeping them the same as for the  $Y_i$  for  $i = 1, \dots, k$  (and making arbitrary choices for  $i = k+1, \dots, m$ ). Then we get a unique extension  $\bar{\mathcal{M}}$  of  $\mathcal{L}'$  to  $\bar{T}$ . Finally Lemma 193 produces the desired extension on  $\bar{S}$ .

The uniqueness follows from Hartog's theorem and the property in the statement of Theorem 194, which defines  $\bar{\mathcal{L}}$  on  $S'$ . The point is that two extensions of  $\mathcal{L}$  which agree outside of a codimension 2 set are equal by Hartog's theorem.  $\square$

#### 14. THE CERESA CYCLE AND THE HAIN-REED BUNDLE

14.1. Fix an integer  $g > 1$  and let  $\mathcal{T}_g$  denote the Teichmüller space of a smooth, projective genus  $g$  Riemann surface  $X$ . Write  $\Gamma_g$  for the mapping class group, the space of orientation preserving diffeomorphisms of  $X$  taken modulo isotopy. Then,  $\mathcal{T}_g$  is a complex manifold, which is isomorphic as a real manifold to  $\mathbb{R}^{6g-6}$ . Moreover,  $\Gamma_g$  acts on  $\mathcal{T}_g$  with orbifold quotient  $\mathcal{M}_g$ , the moduli stack of smooth, projective genus  $g$  curves. Write  $H = H_1(X, \mathbb{Z})$  and write  $Q : \wedge^2 H \rightarrow \mathbb{Z}$  for the intersection pairing. Fix a basis  $e_1, \dots, e_g, f_1, \dots, f_g$  for  $H$  with the property that

$$Q(e_i, f_j) = \delta_{ij}, Q(e_i, e_j) = Q(f_i, f_j) = 0$$

for  $1 \leq i, j \leq g$ . Write  $\mathbf{Sp}_{2g}(\mathbb{Z}) = \mathbf{Sp}(H)$  for the group of automorphisms of  $H$  with determinant one preserving  $Q$ . The action of an element of  $\Gamma_g$  on  $H$  determines a surjection  $\Gamma_g \rightarrow \mathbf{Sp}_{2g}(\mathbb{Z})$ . The kernel  $T_g$  of this surjection is called the Torelli group.

The pairing  $Q$  induces pairings  $Q_k : \wedge^k H \otimes \wedge^k H \rightarrow \mathbb{Z}$  for all non-negative integers  $k$ . These are  $\mathbf{Sp}_{2g}(\mathbb{Z})$ -equivariant and  $(-1)^k$ -symmetric. If we set  $\theta := \sum_{i=1}^g e_i \wedge f_i \in \wedge^2 H$ , then  $\theta$  is  $\mathbf{Sp}_{2g}(\mathbb{Z})$ -invariant. It follows that the maps  $u : \wedge^k H \rightarrow \wedge^{k+2} H$  induced by  $v \mapsto v \wedge \theta$  are  $\mathbf{Sp}_{2g}(\mathbb{Z})$ -equivariant as well.

14.2. **Boundary components of  $\bar{\mathcal{M}}_g$ .** Suppose  $g > 2$  and  $h$  is an integer such that  $1 \leq h \leq \lfloor g/2 \rfloor$ . Then  $D_h$  denotes the Zariski closure of the locus of stable curves consisting of a smooth curve of genus  $h$  and another smooth curve of genus  $g-h$  meeting at one point.  $D_0$  denotes the Zariski closure of the locus of stable curves consisting of a curve of geometric genus  $g-1$  with one node. Then  $D = D_0 \cup \dots \cup D_{\lfloor g/2 \rfloor}$  is a normal crossing divisor whose support is the complement of  $\mathcal{M}_g$  in  $\bar{\mathcal{M}}_g$  (the Deligne-Mumford compactification of  $\mathcal{M}_g$ ).

The divisor  $D_0$  intersects itself in components which we will label as  $D_{0,h}$  for  $0 \leq h \leq \lfloor g/2 \rfloor$ . ( $D$  is not a strict normal crossing divisor.) For  $h > 0$ , the component  $D_{0,h}$  is the Zariski closure of the locus of stable curves consisting of two smooth curves of genus  $h$  and  $g-h-1$  respectively meeting at two points. The generic point of the component  $D_{0,0}$  is a curve of geometric genus  $g-2$  with 2 nodes.

**14.3. Dehn twists and bounding pairs.** If  $\gamma$  is any simple closed curve, we let  $T_\gamma$  denote the Dehn twist of  $X$  determined by  $\gamma$ . This is an element of the mapping class group.

A simple closed curve  $\gamma$  in  $X$  is said to be *bounding* if  $X \setminus \gamma$  is a union of two open Riemann surfaces. A pair  $(\gamma, \delta)$  of homologous simple closed curves, which are not homologically trivial, is said to be a *bounding pair* if  $\gamma$  and  $\delta$  are disjoint, homologous and not homologically trivial. (See Johnson [23].) Contracting the simple closed curves  $\gamma$  and  $\delta$  in a bounding pair to two distinct points produces a curve  $C$  which is a union of two smooth curves of genus  $h =: h(\gamma, \delta)$  and  $g - h - 1$  respectively for  $1 \leq h \leq g - 1$  meeting at two points. Thus contracting the simple closed curves produces a curve in the interior of  $D_{0,h}$ . It is possible to pick the symplectic basis from §14.1 in such a way that  $\{e_1, \dots, e_h, f_1, \dots, f_h\}$  and  $\{e_{h+2}, \dots, e_g, f_{h+2}, \dots, f_g\}$  are symplectic bases for the cohomology of the two components. We say that such a symplectic basis is *adapted to the bounding pair*.

**14.4. Johnson Homomorphism.** The lattice  $\wedge^3 H$  decomposes as a sum of two sublattices as follows [See [19] or [23]]: Let  $u : H \rightarrow \wedge^3 H$  and  $c : \wedge^3 H \rightarrow H$  be the  $\mathbf{Sp}(H)$ -equivariant maps given by

$$u(x) = \theta \wedge x, \quad c(x \wedge y \wedge z) = Q(x, y)z + Q(y, z)x + Q(z, x)y$$

Direct computation shows  $c \circ u(x) = (g - 1)x$ . Define  $I : \wedge^3 H \rightarrow \wedge^3 H$  by the rule

$$I(\omega) = (g - 1)\omega - u \circ c(\omega)$$

**Lemma 195.**  $\ker(I) = \text{im}(u)$ . Moreover, there exists a subgroup  $L$  of  $\wedge^3 H$  such that  $\wedge^3 H = u(H) \oplus L$ .

*Proof.* The assertion that  $\text{im } u \subset \ker I$  follows from the fact that  $c \circ u(x) = (g - 1)x$ . To see that  $\ker I \subset \text{im } u$ , suppose  $\omega \in \ker I$ . Then  $(g - 1)\omega = u \circ c(\omega)$ . So  $(g - 1)\omega \in \text{im } u$ . So it suffices to show that  $\text{im } u$  has a complement,  $L$ , in  $\wedge^3 H$ .

The required subgroup  $L$  is generated by the following elements:

- (i) all products of the form  $v_i \wedge v_j \wedge v_k$  where each  $v_l$  is either  $e_l$  or  $f_l$  and  $i < j < k$ ,
- (ii) all products of the form  $v_i \wedge e_j \wedge f_j$  where  $v_i$  is either  $e_i$  or  $f_i$  and  $i - j$  is not congruent to 0 or 1 modulo  $g$ .

□

Define  $V = \wedge^3 H / \text{im}(u)$ . By the previous lemma, the quotient map  $\wedge^3 H \rightarrow V$  restricts to an isomorphism  $I(\wedge^3 H) \rightarrow V$ . Let  $j : V \rightarrow I(\wedge^3 H)$  denote the inverse isomorphism. We note that, since  $V \cong L$ ,  $V$  is torsion free.

**Theorem 196 (Johnson).** Suppose  $g > 1$ . There is a surjective group homomorphism  $\tau : T_g \rightarrow V$ . If  $(\gamma, \delta)$  is a bounding pair then there is a symplectic basis adapted to  $(\gamma, \delta)$  such that

$$\tau(T_\gamma T_\delta^{-1}) = [(\sum_{i=1}^h e_i \wedge f_i) \wedge f_{h+1}] \in V$$

where  $h = h(\gamma, \delta)$ .

**14.5. The Variation  $V$ .** By abuse of notation, we can view  $H$  as a variation of Hodge structure of weight  $-1$  on  $\mathcal{M}_g$ . We get an exact sequence of variations Hodge structure of weight  $-1$

$$0 \rightarrow H \xrightarrow{u} (\wedge^3 H)(-1) \rightarrow V \rightarrow 0 \quad (197)$$

where  $V \cong \text{im}(I)(-1)$  as in the previous section.

Suppose now that  $C \in D_{0,h}$  is a curve obtained by contracting a bounding pair  $(\gamma, \delta)$  as in §14.3. Since  $\gamma$  and  $\delta$  are homotopic,  $T_\gamma$  and  $T_\delta$  act identically on  $H$  and, thus, on  $V$ . The action of  $T = T_\gamma$

on  $H$  is given by  $h \mapsto h + Q(h, \gamma)\gamma$ . So  $T = \text{id} + N$  is unipotent with monodromy logarithm  $N$  given by  $h \mapsto Q(h, \gamma)\gamma$ . We have  $N^2 = 0$ .

We can find a polydisk  $P = \Delta^{3g-3}$  and an étale map  $j : P \rightarrow \overline{\mathcal{M}}_g$  such that  $j(0) = C$  and  $P' := j^{-1}\mathcal{M}_g \cong \Delta^{*2} \times \Delta^{3g-5}$ . If  $C$  has no automorphisms, then (by shrinking  $P$  if necessary) we can arrange it so that  $j$  is an isomorphism onto its image. Then the monodromy action of  $\mathbb{Z}^2 = \pi_1(P')$  on the pullback of the universal curve to  $P'$  is given  $(a, b) \mapsto T_\gamma^a T_\delta^b$ . In particular, if we write  $N_1$  and  $N_2$  for the logarithms of the monodromy on  $H$ , or, rather, its pullback to  $P'$ , we see that  $N_1 = N_2$ . Moreover, if  $N = N_1$ , then  $N^2 = 0$ .

**Corollary 198.** *We have  $\text{IH}^1(P, V) = NV$ .*

#### 14.6. Normal functions on $\mathcal{M}_g$ .

**Theorem 199.** [20] *There is an element  $\xi \in H^1(\Gamma_g, V)$ , which is the class of a normal function  $v \in \text{ANF}(\mathcal{M}_g, V)_{\mathcal{M}_g}$ . The restriction of  $\xi$  to  $T_g$  under the map*

$$H^1(\Gamma_g, V) \rightarrow H^1(T_g, V) = \text{Hom}(T_g, V)$$

*is twice the Johnson homomorphism.*

**Theorem 200.** *Suppose  $C \in \overline{\mathcal{M}}_g$  is a curve without automorphism obtained by contracting a bounding pair  $(\gamma, \delta)$  as in §14.3. Then, in terms of a symplectic basis adapted to the bounding pair, we have*

$$\text{sing}_C \xi = [2(\sum_{i=1}^h e_i \wedge f_i) \wedge f_{h+1}] \in NV.$$

*Proof.* Consider the commutative diagram

$$\begin{array}{ccc} \text{IH}^1(\mathcal{M}_g, V) & & \\ \downarrow & \searrow & \\ \text{IH}^1(P, V) & & H^1(T_g, V) = \text{Hom}(T_g, V) \\ \downarrow & & \downarrow \\ H^1(\mathbb{Z}^2, V) = H^1(P', V) & \longrightarrow & H^1(\mathbb{Z}, V) = V. \end{array}$$

The map along the bottom row is induced by the map  $\mathbb{Z} \rightarrow \mathbb{Z}^2 = \pi_1(P')$  sending the generator to  $(1, -1)$ . The map on the right is induced by the homomorphism  $\mathbb{Z} \rightarrow T_g$  which sends the generator to  $T_\gamma T_\delta^{-1}$ .

By Theorem (199), the image of  $\xi$  in  $\text{Hom}(T_g, V)$  is the homomorphism which takes  $T_\gamma T_\delta^{-1}$  to  $[2(\sum_{i=1}^h e_i \wedge f_i) \wedge f_{h+1}]$ . By restriction, the image of the Johnson homomorphism in  $H^1(\mathbb{Z}, V) = V$  is precisely  $[2(\sum_{i=1}^h e_i \wedge f_i) \wedge f_{h+1}]$ . The result then follows from Lemma 150.  $\square$

The variation  $V$  has a polarization  $q : V \otimes V \rightarrow \mathbb{Z}(1)$  defined by

$$q(u, v) := \frac{1}{g-1} Q(j(u), j(v)).$$

(See [19, p. 203].) This polarization gives an isomorphism  $a_q : V \rightarrow V^\vee$  and thus a normal function  $v^\vee = a_q(v) \in \text{ANF}(\mathcal{M}_g, V^\vee)_{\overline{\mathcal{M}}_g}$ . This in turn gives a metrized line bundle  $\mathcal{L} := \mathcal{L}(v, v^\vee)$  on  $\mathcal{M}_g$ .

**Theorem 201.** *Suppose  $C$  is a generic curve in  $D_{0,h}$ . Then*

$$h_q(\text{sing}_C \xi, \text{sing}_C \xi) = \frac{4t_1 t_2}{t_1 + t_2} (g - h - 1)h$$

*Proof.* To simplify the notation below, we let  $h' = g - h - 1$ , and recall that

$$\text{sing}_C \zeta = 2 \left[ \sum_{i=1}^h e_i \wedge f_i \wedge f_{h+1} \right] = 2N \left[ \sum_{i=1}^h e_i \wedge f_i \wedge e_{h+1} \right]$$

where  $N = N_1 = N_2$  is the monodromy logarithm around the two branches of  $D_0$  intersecting at  $C$ .

Invoking Proposition (151), it follows that:

$$\begin{aligned} h_q(\text{sing}_C \zeta, \text{sing}_C \zeta) &= \frac{t_1 t_2}{t_1 + t_2} q(2 \left[ \sum_{i=1}^h e_i \wedge f_i \wedge e_{h+1} \right], 2 \left[ \sum_{i=1}^h e_i \wedge f_i \wedge f_{h+1} \right]) \\ &= \frac{4t_1 t_2}{(g-1)(t_1 + t_2)} Q(j \left[ \left( \sum_{i=1}^h e_i \wedge f_i \right) \wedge e_{h+1} \right], j \left[ \left( \sum_{i=1}^h e_i \wedge f_i \right) \wedge f_{h+1} \right]). \end{aligned}$$

Now, for  $v = e_{h+1}$  or  $f_{h+1}$ ,  $c((\sum_{i=1}^h e_i \wedge f_i) \wedge v) = h\theta \wedge v$ . So using this to compute  $j$ , we see that

$$\begin{aligned} h_q(\text{sing}_C \zeta, \text{sing}_C \zeta) &= \frac{4t_1 t_2}{(g-1)(t_1 + t_2)} Q \left( h' \sum_{i=1}^h e_i \wedge f_i \wedge e_{h+1} - h \sum_{i=h+2}^g e_i \wedge f_i \wedge e_{h+1}, \right. \\ &\quad \left. h' \sum_{i=1}^h e_i \wedge f_i \wedge f_{h+1} - h \sum_{i=h+2}^g e_i \wedge f_i \wedge f_{h+1} \right) \\ &= \frac{4t_1 t_2}{(g-1)(t_1 + t_2)} [(h')^2 h + h^2 (h')] \\ &= \frac{4t_1 t_2}{(g-1)t_1 + t_2} (h')h(h + h') \\ &= \frac{4t_1 t_2}{t_1 + t_2} (g - h - 1)h \end{aligned}$$

since  $h + h' = g - 1$ . □

## 15. GENERAL JUMP PAIRING

**15.1. General Pairing.** The goal of this section is to give a definition of the jump pairing on local intersection cohomology without the assumption that the boundary divisor is normal crossing.

To this end, we fix our usual notation that  $\mathcal{H}$  is a variation of pure Hodge structure with  $\mathbb{Q}$  coefficients on a complex manifold  $S$  which is a Zariski open subset of another manifold  $\bar{S}$ . Write  $j : S \rightarrow \bar{S}$  for the embedding. Since we will only be concerned with the local situation, we will assume that  $\bar{S} = \Delta^r$  is a polydisk.

The intermediate extension  $j_{!*} \mathcal{H}_{\mathbb{Q}}[r]$  gives rise to a perverse sheaf  $\text{IC}(\mathcal{H})$  on  $\bar{S}$ . Write  $\mathcal{P}_{\bar{S}} \mathcal{P}$  for the set of all isomorphism classes mixed extensions of  $\text{IC}(\mathbb{Q}_{\bar{S}})$  by  $\text{IC}(\mathcal{H})$  by  $\text{IC}(\mathbb{Q}_{\bar{S}})$  in the category of perverse sheaves on  $\bar{S}$ . For a local system of  $\mathbb{Q}$ -vector spaces  $\mathcal{L}$  on  $S$ , we can write  $\text{IH}^k(\mathcal{L})$  for  $\text{IH}_0^k(\mathcal{L})$ . By shrinking  $\bar{S}$  if necessary, we can assume that  $\text{IH}_0^k(\mathcal{L}) = \text{IH}^k(\bar{S}, \mathcal{L})$ .

We get a map

$$\pi : \mathcal{P} \rightarrow \text{IH}^1(\mathcal{H}_{\mathbb{Q}}) \times \text{IH}^1(\mathcal{H}_{\mathbb{Q}}^{\vee})$$

by identifying the intersection cohomology groups with the appropriate extension groups. Moreover, since  $\text{IH}^1(\bar{S}, \mathbb{Q}) = \text{IH}^2(\bar{S}, \mathbb{Q}) = 0$ , Proposition 166 shows that  $\pi$  is surjective and, for any pair

$$(\alpha, \beta) \in \text{IH}^1(\mathcal{H}_{\mathbb{Q}}) \times \text{IH}^1(\mathcal{H}_{\mathbb{Q}}^{\vee}), \tag{202}$$

the set  $\text{Extpan}(\alpha, \beta)$  injects into  $\mathcal{P}$ . It follows that  $\mathcal{P}$  has the structure of a biextension of  $\text{IH}^1(\mathcal{H}_{\mathbb{Q}}) \times \text{IH}^1(\mathcal{H}_{\mathbb{Q}}^{\vee})$  by  $\text{IH}^1(\bar{S}, \mathbb{Q}) = 0$ .

**Corollary 203.** *For each pair  $(\alpha, \beta)$  as in (202), there exists a unique element  $X = X(\alpha, \beta)$  of  $\mathcal{P}$  with  $\pi(X) = (\alpha, \beta)$ .*

**Proposition 204.** *Suppose  $(\bar{\varphi}, \varphi) : (\Delta, \Delta^*) \rightarrow (\bar{S}, S)$  is a test curve. Then  $\varphi^* X(\alpha, \beta)$  is in the biextension  $\mathcal{R}_{\mathbb{Q}}$  of (179). In other words, both  $\varphi^* \alpha$  and  $\varphi^* \beta$  vanish.*

*Proof.* This follows directly from Theorem 111.  $\square$

**Definition 205.** The jump  $j(\alpha, \beta, \bar{\varphi})$  of  $\alpha$  and  $\beta$  along a test curve  $\bar{\varphi}$  is the number  $\tilde{\tau} \varphi^* X(\alpha, \beta)$ .

Let  $\mathcal{Q}_S$  (or simply  $\mathcal{Q}$  if  $S$  is clear) denote the set of isomorphisms of mixed extensions of  $\mathbb{Q}$  by  $\mathcal{H}$  by  $\mathbb{Q}$  on  $S$  such that the associated classes  $\alpha \in H^1(S, \mathcal{H})$  and  $\beta \in H^1(S, \mathcal{H}^{\vee})$  lie in  $\text{IH}^1(\mathcal{H})$  and  $\text{IH}^1(\mathcal{H}^{\vee})$  respectively. Here we view  $\alpha$  and  $\beta$  as extension classes (of  $\mathbb{Q}$  by  $\mathcal{H}$  and of  $\mathcal{H}$  by  $\mathbb{Q}$  respectively).

**Lemma 206.** *For  $(\alpha, \beta) \in \text{IH}^1(\mathcal{H}) \times \text{IH}^1(\mathcal{H}^{\vee})$ ,  $\text{Extpan}(\alpha, \beta)$  is the fiber of  $\mathcal{Q}$  over  $(\alpha, \beta)$ .*

*Proof.* We need to show that  $G(f_0, f_1) = 0$  for every pair  $(f_0, f_1) \in \text{Hom}(\mathbb{Q}, \mathcal{H}) \times \text{Hom}(\mathcal{H}, \mathbb{Q})$ . By symmetry and additivity of  $G(f_0, f_1)$ , it suffices to show that the composition

$$\text{Hom}(\mathcal{H}, \mathbb{Q}) \times \text{IH}^1(\mathcal{H}) \rightarrow \text{Ext}^1(\mathbb{Q}, \mathbb{Q}) = H^1(S, \mathbb{Q})$$

induced by the cup product vanishes. (Here the Hom and Ext groups are taken in the category of local systems on  $S$ .)

To see this, note that  $\text{Hom}(\mathcal{H}, \mathbb{Q}) = \text{Hom}(\text{IC}(\mathcal{H}), \text{IC}(\mathbb{Q}))$  (by restriction). So we get a commutative diagram

$$\begin{array}{ccc} \text{Hom}_S(\text{IC}(\mathcal{H}), \text{IC}(\mathbb{Q})) \times \text{Ext}_S^1(\text{IC}(\mathcal{H}), \text{IC}(\mathbb{Q})) & \longrightarrow & \text{Ext}_S^1(\text{IC}(\mathbb{Q}), \text{IC}(\mathbb{Q})) = H^1(\bar{S}, \mathbb{Q}) = 0 \\ \parallel & & \downarrow \\ \text{Hom}_S(\mathcal{H}, \mathbb{Q}) \times \text{IH}^1(\mathcal{H}) & \longrightarrow & \text{Ext}_S^1(\mathbb{Q}, \mathbb{Q}) = H^1(S, \mathbb{Q}) \end{array}$$

where the downward arrows are restriction. The result follows immediately.  $\square$

**Corollary 207.** *The map  $\mathcal{Q} \rightarrow \text{IH}^1(\mathcal{H}) \times \text{IH}^1(\mathcal{H}^{\vee})$  makes  $\mathcal{Q}$  into a biextension of  $\text{IH}^1(\mathcal{H}) \times \text{IH}^1(\mathcal{H}^{\vee})$  by  $H^1(S, \mathbb{Q})$ . Moreover, the map  $\mathcal{P} \rightarrow \mathcal{Q}$  induced by restriction is a morphism of biextensions.*

*Explanation.* Here the operations  $+_1$  and  $+_2$  are the obvious ones coming from §10.5 as is the action of  $H^1(S, \mathbb{Q}) = \text{Ext}_S^1(\mathbb{Q}, \mathbb{Q})$ .  $\square$

Now suppose  $\bar{\varphi} : \Delta \rightarrow \bar{S}$  is a test curve. Since the classes  $\alpha$  and  $\beta$  vanish on restriction to  $\Delta^*$  via  $\varphi$ , we get a homomorphism of biextensions  $\varphi^* : \mathcal{Q}_S \rightarrow \mathcal{Q}_{\Delta^*}$ . Via the isomorphism  $\tilde{\tau} : \mathcal{Q}_{\Delta^*} \rightarrow \mathbb{Q}$ , we then get a number  $\tilde{\tau} \varphi^* X$  for any isomorphism class  $X \in \mathcal{Q}$ .

**15.2. Comparison with the asymptotic height pairing.** Now, we want to compare the general jump pairing from Definition 205 with the asymptotic height pairing defined earlier. To do this, we first want to generalize Theorem 153 to the local systems  $X \in \mathcal{Q}_S$  where  $S = \Delta^{*r}$ . Pick  $a \in \Delta^*$ . Then, for each  $r$ -tuple,  $t = (t_1, \dots, t_r) \in \mathbb{Z}_{\geq 0}$ , we can write  $\bar{\varphi}_t : \Delta \rightarrow \Delta^r$  for the test curve  $s \mapsto a(s^{t_1}, \dots, s^{t_r})$ . Write

$$\tilde{\tau}_t X := \tilde{\tau} \varphi^* X.$$

It is easy to see that this rational number does not depend on the choice of  $a$ . Recall that  $\epsilon_i = (0, \dots, 0, 1, \dots, 0)$  with the 1 in the  $i$ -th place.

For any non-negative integer  $r$ , write  $\text{Perv}(\Delta^r)$  for the category of perverse sheaves on  $\Delta^r$  and, following Saito's notation from [33, §3.1], write  $\text{Perv}(\mathbb{Q}_X)_{nc}$  for the full subcategory consisting of perverse sheaves which are constructible with respect to the stratification induced by the coordinate hyperplanes.

**Lemma 208.** *Suppose  $a$  and  $b$  are non-negative integers with  $a + b = r$ ,  $y = (y_1, \dots, y_a) \in (\Delta^*)^a$  and write  $\bar{\varphi} : \Delta^b \rightarrow \Delta^r$  for the map sending  $z = (z_1, \dots, z_b)$  to  $(y_1, \dots, y_a, z_1, \dots, z_b)$ . If  $\mathcal{F}$  is a perverse sheaf in  $\text{Perv}(\Delta^r)_{nc}$  then  $\bar{\varphi}^* \mathcal{F}$  is an object in  $\text{Perv}(\Delta^b)_{nc}$ . Consequently the functor  $\bar{\varphi}^* : \text{Perv}(\Delta^r)_{nc} \rightarrow \text{Perv}(\Delta^b)_{nc}$  is exact.*

*Proof.* As in [33, §3.1], the perverse sheaves in  $\text{Perv}(\Delta^r)_{nc}$  consist of perverse sheaves on  $\Delta^r$  with characteristic variety contained in the conormal bundles of the intersections of the coordinate hyperplanes. Consequently, the map  $\bar{\varphi} : \Delta^b \rightarrow \Delta^r$  is non-characteristic. The result then follows from Kashiwara's theorem on non-characteristic restriction.  $\square$

**Corollary 209.** *We have  $\tilde{\tau}_{\epsilon_i} X(\alpha, \beta) = 0$  for all integers  $i$  with  $1 \leq i \leq n$ .*

*Proof.* Pick  $a \in \Delta^*$  to define  $\bar{\varphi}_t : \Delta \rightarrow \Delta^r$  as above for any  $t \in \mathbb{Z}_{\geq 0}$ . Since  $\bar{\varphi}_{\epsilon_i}^* : \text{Perv}(\Delta^r)_{nc} \rightarrow \text{Perv}(\Delta)_{nc}$  is exact,  $\varphi_{\epsilon_i}^* X(\alpha, \beta)$  is in the trivial biextension  $\mathcal{P}_\Delta$ . So  $\tilde{\tau}_{\varphi_{\epsilon_i}^*} X(\alpha, \beta) = 0$ .  $\square$

**Corollary 210.** *Suppose  $X \in \mathcal{Q}_{\Delta^{*r}}$ , and  $t \in \mathbb{Z}_{\geq 0}^r$ . Then*

$$\tilde{\tau}_t X = \tilde{\tau}_t X(\alpha, \beta) + \sum_{i=1}^r \tilde{\tau}_{\epsilon_i} X.$$

*Proof.* We have  $X = X(\alpha, \beta) + E$  for some element  $E \in \text{Ext}_{\Delta^{*r}}^1(\mathbb{Q}, \mathbb{Q}) = H^1(\Delta^{*r}, \mathbb{Q}) = \mathbb{Q}^r$ . Consequently,  $\tilde{\tau}_t X = \tilde{\tau}_t X(\alpha, \beta) + \tilde{\tau}_t E$ . The result follows from the (easy) fact that  $\tilde{\tau}_t E = \sum \tilde{\tau}_{\epsilon_i}(E) t_i$ .  $\square$

We now state an analogue of Theorem 153.

**Proposition 211.** *Suppose  $X \in \mathcal{Q}_{\Delta^{*r}}$  with  $\pi(X) = (\alpha, \beta)$ . Then*

$$h(t)(\alpha, \beta) = -\tilde{\tau}_t(X) + \sum_{i=1}^r \tilde{\tau}_{\epsilon_i} t_i.$$

*Proof.* The proof is essentially the same as the proof of Theorem 153.  $\square$

**Theorem 212.** *Suppose  $\mathcal{H}$  is a torsion free variation of pure Hodge structure on  $(\Delta^*)^r$  and  $(\alpha, \beta) \in \text{IH}^1(\mathcal{H}) \times \text{IH}^1(\mathcal{H}^\vee)$ . Let  $(t_1, \dots, t_r) \in \mathbb{Z}_{\geq 0}^r$ , and write  $\bar{\varphi} : \Delta \rightarrow \Delta^r$  for the test curve  $s \mapsto (s^{t_1}, \dots, s^{t_r})$ . Then*

$$h(\alpha, \beta)(t) = -j(\alpha, \beta, \bar{\varphi}_t).$$

*Proof.* Apply Proposition 211 to  $X = X(\alpha, \beta)$  using the fact that  $\tilde{\tau}_{\epsilon_i} X = 0$  for all  $i$ .  $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD USA

*E-mail address:* `pbroshanmath.umd.edu`

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX USA

*E-mail address:* `gpearlmath.tamu.edu`